Symbolic Derivation of a Probability-Ready Expression for the Reliability Analysis of a Multi-State Delivery Network

Ali Muhammad Ali Rushdi[1*] and Motaz Hussain Amashah[1]

[1]*Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, P.O.Box 80204, Jeddah 21589, Kingdom of Saudi Arabia.*

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Abstract

This paper deals with the reliability of a multi-state delivery network (MSDN) with multiple suppliers, transfer stations and markets (depicted as vertices), connected by branches of multi-state capacities, delivering a certain commodity or service between their end vertices. We utilize a symbolic logic expression of the network success to satisfy the market demand within budget and production capacity limitations even when subject to deterioration. This system success is a two-valued function expressed in terms of multi-valued component successes, and it has been obtained in the literature in minimal form as the disjunction of prime implicants or minimal paths of the pertinent network. The main contribution of this paper is to provide a systematic procedure for converting this minimal expression into a probability-ready expression (PRE). We successfully extrapolate the PRE concept from the two-valued logical domain to the multi-valued logical domain. This concept is of paramount importance since it allows a direct transformation of a random logical expression, on a one-to-one one, to its statistical expectation form, simply by replacing all logic variables by their statistical expectations, and also substituting arithmetic multiplication and addition for their logical counterparts (ANDing and ORing). The statistical expectation of the expression is its probability of being equal to 1, and is simply called network reliability. The proposed method is illustrated with a detailed symbolic example of a real-case study, and it produces a more precise version of the same numerical value that was obtained earlier by an alternative means. This paper is a part of an ongoing activity to develop pedagogical material for various candidate techniques for assessing multi-state reliability.

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*Corresponding author: E-mail: arushdi@kau.edu.sa, arushdi@ieee.org, alirushdi@gmail.com;

1 Introduction

Consider a multi-state delivery network (MSDN) with multiple suppliers, in which a vertex denotes a supplier, a transfer station or a market, while a branch denotes a carrier providing the delivery service for a pair of vertices [1-5]. The capacity that is available for a specific customer of the carrier responsible for the delivery on a branch is treated as a multi-state variable, since this capacity is shared among several customers including the one under consideration. The addressed problem is to evaluate the network reliability, the probability that the MSDN with the deterioration consideration can satisfy the market demand within the budget and production capacity limitations. Lin et al. [1] developed an algorithm to deduce the binary system success in terms of the multi-valued component successes, and then to transform this success to its expectation, which is called network reliability. They also provided a numerical example to illustrate the first part of their solution procedure. However, they did not explain the details of the second part of this procedure, which pertains to the replacement of the logical expression of a multi-state disjunctive normal form (DNF) by its expectation. In fact, they utilized an automated implementation of the method of the recursive sum of disjoint products (RSDP) [6,7], which is one of many candidate methods [1-21], characterized as being more computationally efficient, especially for larger networks [10].

This paper makes its point through the multi-valued symbolic analysis of the aforementioned specific (albeit standard) problem of a multi-state delivery network of a supply chain [1]. Our results provide a truly independent means to check and verify the earlier solution of this problem reported in [1]. The present analysis can be extended to other multi-state systems (MSSs) of comparable sizes, and might be automated to handle more general MSSs that are of larger sizes. The paper utilizes algebraic techniques of multiple-valued logic to evaluate the system binary output or success as a function of the system multi-valued inputs. The formula for this output is written as a probability–ready expression, thereby allowing its immediate conversion, on a one-to-one basis, into a probability or expected value. The paper strives to provide a pedagogically-insightful paradigm that establishes a clear and fruitful interrelationship between binary modeling and MSS modeling by stressing that multi-valued concepts are natural and simple extensions of two-valued ones. The mathematical treatment and details given herein strictly adhere to this paradigm, and hence assert the notion that there is no need to reinvent the wheel while extrapolating from two states to multiple states. We hope that our work might help researchers avoid the waste of time and duplication of effort involved in trying to handle multi-state problems as if they were totally new.

The organization of the remainder of this paper is as follows. Section 2 presents important pertinent assumptions and notation. Section 3 introduces the running example of a multi-state delivery network (MSDN) with multiple suppliers, borrowed from [1]. Section 4 extends the concept of a probability-ready expression (PRE) from the binary to the multi-state case. Section 5 provides a detailed symbolic conversion of the given multi-state sum-of-products expression of system success for the running example into a PRE form, and hence computes a more precise version of the same numerical value that was obtained earlier in [1]. Section 5 also identifies techniques for validating and checking the results obtained. Section 6 concludes the paper and suggests some future work.

2 Assumptions and Notation

2.1 Assumptions

- The model considered is one of a system with binary output and multistate components, specified by the structure or success function $S(\boldsymbol{X})$ [7,22]

$$S: \{0, 1, \cdots, m_1\} \times \{0, 1, \cdots, m_2\} \times \ldots \times \{0, 1, \cdots, m_n\} \to \{0, 1\}. \qquad (1)$$

- The system is generally non-homogeneous, i.e., the number of system states (two) and the numbers of component states $(m_1 + 1), (m_2 + 1), \cdots, (m_n + 1)$ might differ. When these numbers have a common value, the system reduces to a homogeneous one.
- The system is a non-repairable one with statistically independent non-identical (heterogeneous) components.
- The system is a coherent one enjoying the properties of causality, monotonicity, and component relevancy [22-28].

2.2 Notation

Chart 1. Description of symbols

Symbol	Description
X_k	A multivalued input variable representing component k $(1 \le k \le n)$, where $X_k \in \{0, 1, \dots, m_k\}$, and $m_k \ge 1$ is the highest value of X_k.
$X_k\{j\}$	A binary variable representing instant j of X_k $X_k\{j\} = \{X_k = j\}$, i.e., $X_k\{j\} = 1$ if $X_k = j$ and $X_k\{j\} = 0$ if $X_k \neq j$. The instances $X_k\{j\}$ for $\{0 \le j \le m_k\}$ form an orthonormal set, namely, for $\{1 \le k \le n\}$ $$\bigvee_{j=0}^{m_k} X_k\{j\} = 1, \tag{2a}$$ $$X_k(j_1)X_k(j_2) = 0 \text{ for } j_1 \neq j_2. \tag{2b}$$ Orthonormality is very useful in constructing inverses or complements. The complement of the union of certain instances is the union of the complementary instances. In particular, the complement of $X_k\{\ge j\} = X_k\{j, j+1, \dots, m_k\}$ is $X_k\{< j\} = X_k\{0, 1, \dots, j-1\}$.
$X_k\{\ge j\}$	An upper value of X_k $\{0 \le j \le m_k\}$: $$X_k\{\ge j\} = X_k\{j, j+1, \dots, m_k\} = \bigvee_{i=j}^{m_k} X_k\{i\} = X_k\{j\} \vee X_k\{j+1\} \vee \dots \vee X_k\{m_k\}. \tag{3}$$ The value $X_k\{\ge 0\}$ is identically 1. The set $X_k\{\ge j\}$ for $\{1 \le j \le m_k\}$ is neither independent nor disjoint, and hence it is difficult to be handled mathematically, but it is very convenient for translating the verbal or map/tabular description of a coherent component into a mathematical form when viewing component success at level j. The complement of $X_k\{\ge j\}$ is $$X_k\{< j\} = X\{0, 1, \dots, j-1\} = X_k\{0\} \vee X_k\{1\} \dots \vee X_k\{j-1\} = X_k\{k \le (j-1)\}. \tag{4}$$
$X_k\{\le j\}$	A lower value of $X_k \{0 \le j \le m_k\}$: $$X_k\{\le j\} = X_k\{0, 1, \dots, j\} = \bigvee_{i=0}^{j} X_k\{i\} = X_k\{0\} \vee X_k\{1\} \dots \vee X_k\{j-1\} \vee X_k\{j\}. \tag{5}$$ The value $X_k\{\le m_k\}$ is identically 1. The set $X_k\{\le j\}$ for $\{0 \le j \le (m_k - 1)\}$ is neither independent nor disjoint, and hence it is not convenient for mathematical manipulation though it is suitable for expressing component failure at level $(j+1)$. Instances, upper values and lower values are related by $$X_k\{j\} = X_k\{\ge j\}X_k\{< (j+1)\} = X_k\{\ge j\}\bar{X}_k\{\ge (j+1)\} = X_k\{\le j\}X_k\{> (j-1)\}$$ $$= X_k\{\le j\}\bar{X}_k\{\le (j-1)\}. \tag{6}$$
S	A binary output variable representing the system, where $S \in \{0, 1\}$. The function $S(\boldsymbol{X})$ is usually called the system success or the structure function. Its complement $\bar{S}(\boldsymbol{X})$ is called system failure, and is also a binary variable. The logical sum and arithmetic sum of success and failure are both equal to 1, namely $$(S(\boldsymbol{X}) \vee \bar{S}(\boldsymbol{X})) = (S(\boldsymbol{X}) + \bar{S}(\boldsymbol{X})) = 1. \tag{7}$$

3 Detailed Running Example

Lin et al. [1] studied the multi-state delivery network (MSDN) with multiple suppliers shown in Fig. 1. The network contains two suppliers, one market, two transfer centers and eight branches. The network has specific data of delivery costs, probability distributions of all branches and available capacities that are listed

in [1], together with the suppliers' production capacities. The minimal paths (MPs) connecting source s_1 and terminal t can be expressed as $P_1 = \{b_1, b_6\}$, $P_2 = \{b_2, b_7\}$ and $P_3 = \{b_2, b_5, b_8\}$, and the MPs connecting source s_2 and terminal t are $P_4 = \{b_3, b_7\}$, $P_5 = \{b_3, b_5, b_8\}$ and $P_6 = \{b_4, b_8\}$. The deterioration rate vector for the six MPs is given together with the demand, production capacity and the budget. The final multi-state success expression derived from Table 2 in [1] (with appropriate translation of notation) is given by

$$
\begin{aligned}
S = {} & X_3\{\geq 3\}X_5\{\geq 3\}\,X_8\{\geq 3\} \\
& \vee X_3\{\geq 3\}X_7\,\{\geq 3\} \\
& \vee X_2\{\geq 3\}X_5\{\geq 3\}\,X_8\{\geq 3\} \\
& \vee X_2\{\geq 2\}X_3\{\geq 2\}\,X_4\{\geq 2\}X_7\{\geq 3\}\,X_8\{\geq 2\} \\
& \vee X_2\{\geq 3\}X_7\{\geq 3\} \\
& \vee X_1\{\geq 2\}X_3\{\geq 2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\} \\
& \vee X_1\{\geq 2\}X_2\{\geq 2\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\} \\
& \vee X_1\{\geq 2\}X_2\{\geq 2\}\,X_3\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 3\}.
\end{aligned}
\tag{8}
$$

The numerical values for the expectations of various variable instances, computed from data given in [1] are listed in Table 1.
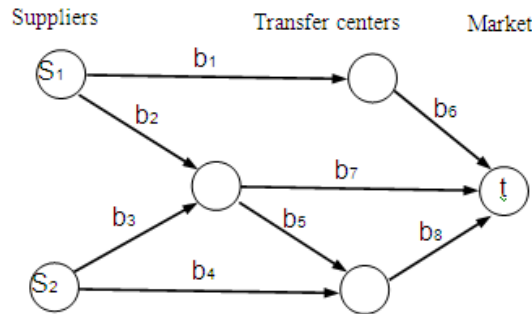
Fig. 1. The multi-state delivery network (MSDN) with multiple suppliers studied in [1]

Table 1. Numerical values for the expectations of various variable instances, computed from data given in [1]

$X_1\{\geq 2\}$	0.897	$X_3\{\geq 3\}$	0.905
$X_2\{\geq 3\}$	0.892	$X_3\{\geq 2\}$	0.953
$X_2\{< 3\}$	0.108	$X_3\{2\}$	0.048
$X_2\{\geq 2\}$	0.965	$X_3\{< 3\}$	0.095
$X_2\{2\}$	0.073	$X_4\{\geq 2\}$	0.863
$X_4\{< 2\}$	0.137	$X_5\{\geq 3\}$	0.903
$X_5\{< 3\}$	0.097	$X_6\{\geq 2\}$	0.943
$X_7\{\geq 2\}$	0.945	$X_7\{\geq 3\}$	0.884
$X_7\{2\}$	0.061	$X_7\{< 3\}$	0.116
$X_8\{\geq 3\}$	0.906	$X_8\{\geq 2\}$	0.965
$X_8\{2\}$	0.059	$X_8\{< 3\}$	0.094

4 Probability-Ready Expressions

The concept of a probability-ready expression (PRE) is well-known in the two-valued logical domain [13,29-36], and it is still valid for the multi-valued logical domain [22-28]. A Probability-Ready Expression is a random expression that can be directly transformed, on a one-to-one basis, to its statistical expectation (its probability of

being equal to 1) by replacing all logic variables by their statistical expectations, and also replacing logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts. A logic expression is a PRE if

    a)   All *ORed* products (i.e., terms formed by ANDing of literals) are *disjoint* (*mutually exclusive*), and
    b)   All *ANDed* sums (i.e., alterms formed via ORing of literals) are *statistically independent*.

Condition (a) is satisfied if for every pair of ORed terms, there is at least a single opposition, i.e., there is at least one variable that appears with a certain set of instances in one term and appears with a complementary set of instances in the other. Condition (b) is satisfied if for every pair of ANDed alterms (sums of disjunctions of literals), one alterm involves variables describing a certain set of components, while the other alterm depends on variables describing a set of different components (under the assumption of independence of components) [31,32,34,36,37].

While there are many methods to introduce characteristic (a) of orthogonality (disjointness) into a Boolean expression [6,13,34,38], there is no way to induce characteristic (b) of statistical independence. The best that one can do is to observe statistical independence when it exists, and then be careful not to destroy or spoil it and take advantage of it. Since one has the freedom of handling a problem from a success or a failure perspective, a choice should be made as to which of the two perspectives can more readily produce a PRE form. It is better to look at success for a system of no or poor redundancy (a series or almost-series system), and to view failure for a system of full or significant redundancy (a parallel or almost- parallel system) [13,31-36]. A prominent method for the creation of a PRE is the Boole-Shannon Expansion [24,34,39-42].

The introduction of orthogonality might be achieved as follows. If neither of the two terms A and B in the sum $(A \lor B)$ subsumes the other ($A \lor B \neq A$ and $A \lor B \neq B$) and the two terms are not disjoint ($A \land B \neq 0$), then B can be disjointed with A by factoring out any common factor (using Boolean quotients [22,34]) and then applying the *Reflection Law*, namely

$$A \lor B = C((A/C) \lor (B/C)) = C((A/C) \lor \overline{(A/C)}(B/C)) = A \lor \overline{(A/C)}B. \tag{9}$$

In (9), the symbol C denotes the common factor of A *and* B, and the Boolean quotient (A/C) might be viewed as the term A with its part common with B removed. If B subsumes A, then $C = A$ and $A/C = 1$, so that $\overline{(A/C)} = 0$, which means that B is absorbed in A. Note that (9) leaves the term A intact and replaces the term B by an expression that is disjoint with A. The quotient (A/C) is a product of e entities Y_k ($1 \leq k \leq e$), so that $\overline{(A/C)}$ might be expressed via *De Morgan's Law* as a disjunction of the form

$$\overline{(A/C)} = \bigvee_{k=1}^{e} \bar{Y}_k. \tag{10}$$

Note that each Y_K is a literal that appears in the product A and does not appear in the product B. It stands for a disjunction of certain instances of some variable $X_{i(k)}$ *and hence* $\bar{Y}_k$ is a disjunction of the complementary instances of the same variable. If we combine (9) with (10), we realize that the term B is replaced by e terms ($e \geq 1$), which are each disjoint with the term A, but are not necessarily disjoint among themselves. Therefore, we replace the *De Morgan's Law* in (10) by a disjoint version of it [34], namely

$$\overline{(A/C)} = \bar{Y}_1 \lor Y_1\bar{Y}_2 \lor Y_1Y_2\bar{Y}_3 \lor \dots \lor Y_1Y_2 \dots Y_{e-1}\bar{Y}_e$$
$$= \bar{Y}_1 \lor Y_1\left(\bar{Y}_2 \lor Y_2(\bar{Y}_3 \lor \dots \dots \lor (\bar{Y}_{e-1} \lor Y_{e-1}\bar{Y}_e) \dots)\right). \tag{11}$$

When (11) is combined with (9), one obtains

$$A \lor B = A \lor (\bar{Y}_1 \lor Y_1\bar{Y}_2 \lor Y_1Y_2\bar{Y}_3 \lor \dots \lor Y_1Y_2 \dots Y_{e-1}\bar{Y}_e)B, \tag{12}$$

where the first term A still remains intact, while the second term B is replaced by e terms which are each disjoint with A and are also disjoint among themselves. This means that one has a choice of either disjointing B with A in

$A \vee B$, or disjointing A with B in $B \vee A$. The usual practice that is likely to yield good results is to order the terms in a given disjunction so that those with fewer literals should appear earlier.

Rushdi [24] presented a simple example of the procedure above by considering the following expression

$$S\{0\} = X_1\{0\} \vee X_2\{0\} \vee X_3\{0\} \vee X_4\{0\}, \tag{13}$$

which is not a PRE, since it has ORed quantities that are not *disjoint*. A PRE version of it might be obtained by using the afore-mentioned disjointing procedure, namely

$$S\{0\} = X_1\{0\} \vee \bar{X}_1\{0\}(X_2\{0\} \vee \bar{X}_2\{0\}(X_3\{0\} \vee \bar{X}_3\{0\}X_4\{0\})). \tag{14}$$

However, a much simpler PRE is obtained by simply complementing (13), namely

$$\bar{S}\{0\} = \bar{X}_1\{0\}\bar{X}_2\{0\}\bar{X}_3\{0\}\bar{X}_4\{0\}. \tag{15}$$

The expression in (15) is a PRE since *ANDed* quantities in it are *statistically independent.* This example illustrates that attaining PRE form is possible not only *via* the implementation of a disjointing procedure, but also through effective utilization of statistical independence, which might be manifested in a particular form of the function and lacking in its complementary form.

The implementation of (12) is aided by a few simplification rules, such as

$$X_k(\geq j_1)X_k(\geq j_2) = X_k(\geq j_2) \qquad \text{for } j_2 \geq j_1, \tag{16a}$$

$$X_k(\leq j_1)X_k(\leq j_2) = X_k(\leq j_2) \qquad \text{for } j_2 \leq j_1, \tag{16b}$$

$$X_k(\geq j_1)X_k(\leq j_2) = X_k(j_1, j_1 + 1, \dots, j_2) \qquad \text{for } j_2 \geq j_1, \tag{16c}$$

$$X_k(\geq j_1)X_k(\leq j_2) = 0 \qquad \text{for } j_2 < j_1, \tag{16d}$$

$$\bar{X}_k\{\geq j\} = X_k\{< j\}, \tag{16e}$$

$$X_k(\geq j_1)X_k(< j_2) = X_k(j_1, j_1 + 1, \dots, j_2 - 1) \qquad \text{for } j_2 > j_1, \tag{16f}$$

$$X_k(\geq j_1)X_k(< j_2) = 0 \qquad \text{for } j_2 \leq j_1, \tag{16g}$$

5 Derivation of a Disjoint Sum-of-Products Expression for the Running Example

We rearrange the terms of S in (8) to obtain

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee X_2\{\geq 3\}X_7\{\geq 3\}$
$\vee X_3\{\geq 3\}X_5\{\geq 3\} X_8\{\geq 3\}$
$\vee X_2\{\geq 3\}X_5\{\geq 3\} X_8\{\geq 3\}$
$\vee X_2\{\geq 2\}X_3\{\geq 2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 2\} X_3\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\}$
$\vee X_1\{\geq 2\}X_3\{\geq 2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 2\} X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\tag{17}$

We now apply (12) repeatedly to disjoint the first product (highlighted in yellow) in (18) to each of the following products as follows

$S =$ $X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}\,(X_3\{< 3\} \vee X_3\{\geq 3\}\,X_7\{< 3\})X_5\{\geq 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 2\}X_3\{\geq 2\}X_3\{< 3\}\,X_4\{\geq 2\}X_7\{\geq 3\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\,X_3\{\geq 2\}X_3\{< 3\}X_6\{\geq 2\}\,X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_3\{\geq 2\}\,(X_3\{< 3\} \vee X_3\{\geq 3\}\,X_7\{< 3\})\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\,(X_3\{< 3\} \vee X_3\{\geq 3\}\,X_7\{< 3\})\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}.$ (18)

Equation (18) is now simplified via (16) to yield

$S =$ $X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{\geq 3\}X_5\{\geq 3\}\,X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 2\}X_3\{2\}\,X_4\{\geq 2\}X_7\{\geq 3\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{2\}X_6\{\geq 2\}\,X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_3\{2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{< 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}.$ (19)

We now apply (12) repeatedly to disjoint the second product (highlighted in yellow) in (20) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}\,X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{\geq 3\}X_5\{\geq 3\}\,X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 2\}X_3\{2\}(X_2\{< 3\} \vee X_2\{\geq 3\}\,X_3\{\geq 3\})\,X_4\{\geq 2\}X_7\{\geq 3\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{2\}(X_2\{< 3\} \vee X_2\{\geq 3\}\,X_3\{\geq 3\})X_6\{\geq 2\}\,X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_3\{2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}(X_2\{< 3\} \vee X_2\{\geq 3\}\,X_3\{\geq 3\} \vee X_2\{\geq 3\}\,X_3\{< 3\}X_7\{< 3\})\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{< 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 2\}(X_2\{< 3\} \vee X_2\{\geq 3\}\,X_7\{< 3\})\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$ (20)

Equation (20) is now simplified via (16) to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{\geq 3\}X_5\{\geq 3\}\,X_7\{< 3\}\,X_8\{\geq 3\}$
$\vee\, X_2\{2\}X_3\{2\}\,X_4\{\geq 2\}X_7\{\geq 3\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_6\{\geq 2\}\,X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_2\{< 3\}X_3\{2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{< 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\,X_4\{\geq 2\}X_6\{\geq 2\}\,X_7\{2\}\,X_8\{\geq 2\}.$ (21)

We continue to disjoint the third product (highlighted in yellow) in (22) with its successors

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\ \ \ \boxed{X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}}$
$\vee X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee X_2\{\geq 3\}\boldsymbol{X_3\{\geq 3\}X_5\{\geq 3\}\ X_7\{< 3\}\ X_8\{\geq 3\}(0)}$
$\vee X_2\{2\}\boldsymbol{X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{2\}\boldsymbol{X_3\{2\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{\geq 3\}}$
$\vee X_1\{\geq 2\}X_2\{< 3\}\boldsymbol{X_3\{2\}}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 3\}\boldsymbol{X_3\{2\}}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\textcolor{red}{(X_5\{< 3\} \vee X_5\{\geq 3\}X_7\{\geq 3\} \vee\ X_5\{\geq 3\}X_7\{< 3\}X_8\{< 3\})}$
$\vee X_1\{\geq 2\}X_2\{2\}\boldsymbol{X_3\{< 3\}}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 3\}\boldsymbol{X_3\{< 3\}}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\textcolor{red}{(X_5\{< 3\} \vee X_5\{\geq 3\}X_7\{\geq 3\} \vee\ X_5\{\geq 3\}X_7\{< 3\}X_8\{< 3\})}$

$\hfill (22)$

Note that the fifth term subsumes the third, and hence is multiplied by 0 to indicate that it is absorbed. Terms highlighted in bold violet red ultimately vanish because they are multiplied by factors orthogonal to them. Equation (22) is now simplified via (16) to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\ \ \ \boxed{X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}}$
$\vee X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
$\vee X_1\{\geq 2\}X_2\{< 3\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\textcolor{red}{X_5\{< 3\}}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\textcolor{red}{X_5\{\geq 3\}}\ X_6\{\geq 2\}\ \textcolor{red}{X_7\{2\}\ X_8\{2\}}$
$\vee X_1\{\geq 2\}X_2\{2\}X_3\{< 3\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\textcolor{red}{X_5\{< 3\}}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\textcolor{red}{X_5\{\geq 3\}}X_6\{\geq 2\}\ \textcolor{red}{X_7\{2\}\ X_8\{2\}}.$

$\hfill (23)$

Next, we apply (12) repeatedly to disjoint the fourth product (highlighted in yellow) in (24) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ \ \ \boxed{X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}}$
$\vee \boldsymbol{X_2\{2\}}X_3\{2\}\ X_4\{\geq 2\}\boldsymbol{X_7\{\geq 3\}}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}\boldsymbol{X_2\{2\}}X_3\{2\}X_6\{\geq 2\}\ \boldsymbol{X_7\{\geq 3\}}$
$\vee X_1\{\geq 2\}\boldsymbol{X_2\{< 3\}}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\textcolor{red}{(X_5\{< 3\} \vee X_5\{\geq 3\}X_7\{\geq 3\} \vee\ X_5\{\geq 3\}X_7\{< 3\}X_8\{< 3\}) \vee\ X_5\{\geq 3\}X_7\{< 3\}X_8\{\geq 3\}X_3\{\geq 3\})}$
$\vee X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}\boldsymbol{X_5\{< 3\}}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee X_1\{\geq 2\}\boldsymbol{X_2\{2\}}X_3\{< 3\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2}\{\geq \boldsymbol{3}\}\boldsymbol{X_3}\{< 3\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$({\color{red}X_5}\{< 3\} \vee {\color{red}X_5\{\geq 3\}X_7\{\geq 3\}} \vee {\color{red}X_5\{\geq 3\}X_7\{< 3\}X_8\{< 3\}})$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}.$ (24)

Equation (24) is now simplified via (16) to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$

$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$

$\vee$ <mark>$X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$</mark>

$\vee\ X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$

$\vee\ X_1\{\geq 2\}X_2\{< 3\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}{\color{red}X_5}\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}{\color{red}X_5}\{\geq 3\}X_6\{\geq 2\}\ {\color{red}X_7\{2\}\ X_8\{2\}}$

$\vee\ X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$

$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{< 3\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\ X_4\{\geq 2\}{\color{red}X_5}\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\ X_4\{\geq 2\}{\color{red}X_5}\{\geq 3\}X_6\{\geq 2\}\ {\color{red}X_7\{2\}\ X_8\{2\}}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}{\color{red}X_3}\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}{\color{red}X_3}\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}.$ (25)

Now, we apply (12) repeatedly to disjoint the fifth product (highlighted in yellow) in (26) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$

$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$

$\vee$ <mark>$X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$</mark>

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2\{2\}X_3\{2\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{\geq 3\}}({\color{red}X_4}\{< 2\} \vee {\color{red}X_4}\{\geq 2\}\ {\color{red}X_8}\{< 2\})$

$\vee\ X_1\{\geq 2\}X_2\{< 3\}\boldsymbol{X_3\{2\}}X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ \boldsymbol{X_8}\{\geq \boldsymbol{2}\}({\color{red}X_2}\{\neq 2\} \vee {\color{red}X_2}\{2\}\ {\color{red}X_7}\{< 3\})$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2}\{\geq \boldsymbol{3}\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2}\{\geq \boldsymbol{3}\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2\{2\}}X_3\{< 3\}\boldsymbol{X_4}\{\geq \boldsymbol{2}\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ \boldsymbol{X_8}\{\geq \boldsymbol{2}\}({\color{red}X_3}\{\neq 2\} \vee {\color{red}X_3}\{2\}\ {\color{red}X_7}\{< 3\})$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2}\{\geq \boldsymbol{3}\}X_3\{< 3\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}\boldsymbol{X_2}\{\geq \boldsymbol{3}\}X_3\{< 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$

$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3}\{\geq \boldsymbol{3}\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{2\}.$ (26)

Equation (26) is now simplified via (16) to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$

$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$

$\vee$ <mark>$X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$</mark>

$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}{\color{red}X_4}\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$

$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}{\color{red}X_4}\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ {\color{red}X_8}\{< 2\}$

∨ $X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
            ∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{< 2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$. 　　　　　　　　　(27)

Now, we discover in (28), (29) and (30) that the sixth, seventh and eight products (respectively highlighted in yellow) are already disjoint to each of their following products. Therefore, we jump immediately to the ninth term, highlight it in yellow and apply (12) repeatedly to it so as to disjoint it in (31) to each of the following products therein.

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
∨ $X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
∨ $X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
∨ $X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
∨ $X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ X_8\{< 2\}$
∨ $X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
            ∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{< 2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$. 　　　　　(28)

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
∨ $X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
∨ $X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
∨ $X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
∨ $X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ X_8\{< 2\}$
∨ $X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
            ∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_3\{\geq 3\}X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{< 2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, \boldsymbol{X_7\{2\}}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, \boldsymbol{X_7\{2\}}\, \boldsymbol{X_8\{2\}}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, \boldsymbol{X_7\{2\}}\, \boldsymbol{X_8\{\geq 2\}}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, \boldsymbol{X_7\{2\}}\, \boldsymbol{X_8\{2\}}.$ (29)

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\, X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\, X_8\{\geq 3\}$
$\vee\, X_2\{2\}X_3\{2\}\, X_4\{\geq 2\}X_7\{\geq 3\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\, X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{< 2\}$
$\vee\, \boxed{X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{2\}}X_3\{2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\qquad\qquad\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{2\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{2\}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}X_4\{\geq 2\}X_5\{< 3\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\, X_4\{\geq 2\}X_5\{\geq 3\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{2\}X_3\{< 2\}}X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{2\}}X_3\{2\}\, X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 2\}}X_3\{\geq 3\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 2\}}\boldsymbol{X_3\{\geq 3\}}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}.$ (30)

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\, X_8\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\, X_8\{\geq 3\}$
$\vee\, X_2\{2\}X_3\{2\}\, X_4\{\geq 2\}X_7\{\geq 3\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\, X_7\{\geq 3\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{< 2\}$
$\vee\, X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}$
$\vee\, \boxed{X_1\{\geq 2\}X_2\{2\}X_3\{2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{2\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{2\}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}X_4\{\geq 2\}X_5\{< 3\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\, X_4\{\geq 2\}X_5\{\geq 3\}\, X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}X_2\{2\}\boldsymbol{X_3\{< 2\}}\, X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}$
$\vee\, \boldsymbol{X_1\{\geq 2\}X_2\{2\}X_3\{2\}\, X_4\{\geq 2\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}}\textcolor{red}{(0)}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}}X_3\{< 3\}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\, X_7\{2\}\, X_8\{2\}.$ (31)

Equation (31) is now simplified by omitting the term multiplied by zero (actually absorbed by the ninth term to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\vee\, X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\, X_8\{\geq 3\}$

$\lor X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
$\lor X_2\{2\}X_3\{2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$
$\lor X_1\{\geq 3\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} X_7\{\geq 3\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\} X_8\{< 2\}$
$\lor X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\lor$ <mark>$X_1\{\geq 2\}X_2\{2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$</mark>
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{< 3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}.$ (32)

Now, we apply (12) repeatedly to disjoint the tenth product (now highlighted in yellow) in (33) to each of the following products as follows

$S = X_3\{\geq 3\}X_7 \{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\lor X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
$\lor X_2\{2\}X_3\{2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} X_7\{\geq 3\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\} X_8\{< 2\}$
$\lor X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor$ <mark>$\boldsymbol{X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}}$</mark>
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}} X_4\{\geq 2\}X_5\{< 3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}} X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}\boldsymbol{X_2\{2\}X_3\{< 2\}}X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\lor \boldsymbol{X_1\{\geq 2\}X_2\{\geq 3\}}X_3\{< 3\}\color{red}{X_3\{\neq 2\}}\color{black}{X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\} X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}} X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\} X_7\{2\} X_8\{2\}.$ (33)

Equation (33) is now simplified to yield

$S = X_3\{\geq 3\}X_7 \{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\lor X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
$\lor X_2\{2\}X_3\{2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} X_7\{\geq 3\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\} X_8\{< 2\}$
$\lor X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor$ <mark>$X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$</mark>
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{< 3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$

$\lor X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} \, X_4\{\geq 2\}X_6\{\geq 2\} \, X_7\{\geq 2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}.$ \hfill (34)

Now, we apply (12) repeatedly to disjoint the eleventh product (now highlighted in yellow) in (35) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\lor X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} \, X_8\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} \, X_8\{\geq 3\}$
$\lor X_2\{2\}X_3\{2\} \, X_4\{\geq 2\}X_7\{\geq 3\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} \, X_7\{\geq 3\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} \, X_7\{\geq 3\} \, X_8\{< 2\}$
$\lor X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} \, X_4\{\geq 2\} \, X_6\{\geq 2\} \, X_7\{\geq 2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\} \, X_4\{\geq 2\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor \quad \boxed{X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{< 3\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{\geq 3\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} \, X_4\{\geq 2\}X_6\{\geq 2\} \, X_7\{\geq 2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 3\}X_3\{\neq 2\}X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}.$ \hfill (35)

Equation (35) is now simplified to yield

$S = X_3\{\geq 3\}X_7\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\lor X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} \, X_8\{\geq 3\}$
$\lor X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} \, X_8\{\geq 3\}$
$\lor X_2\{2\}X_3\{2\} \, X_4\{\geq 2\}X_7\{\geq 3\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} \, X_7\{\geq 3\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} \, X_7\{\geq 3\} \, X_8\{< 2\}$
$\lor X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} \, X_4\{\geq 2\} \, X_6\{\geq 2\} \, X_7\{\geq 2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{2\} \, X_4\{\geq 2\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor \quad \boxed{X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{< 3\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{\geq 3\} \, X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} \, X_4\{\geq 2\}X_6\{\geq 2\} \, X_7\{\geq 2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{\geq 2\}$
$\lor X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} \, X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} \, X_7\{2\} \, X_8\{2\}.$ \hfill (36)

Now, we apply (12) repeatedly to disjoint the twelfth product (now highlighted in yellow) in (37) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\{\geq 3\}$

$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ X_8\{< 2\}$
$\vee\ X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ \;\boxed{X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}}$
$\vee\ X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}\boldsymbol{X_3\{< 2\}}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}\boldsymbol{X_3\{< 2\}}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}\boldsymbol{X_3\{< 2\}}\ X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ \boldsymbol{X_1\{\geq 2\}}X_2\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}\boldsymbol{X_5\{< 3\}}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}\boldsymbol{(0)}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}.$ (37)

Equation (37) is now simplified to yield

$S = X_3\{\geq 3\}X_7\ \{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ X_8\{< 2\}$
$\vee\ X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ \;\boxed{X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}}$
$\vee\ X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{< 2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}.$ (38)

Now, we apply (12) repeatedly to disjoint the twelfth product (now highlighted in yellow) in (39) to each of the following products as follows

$S = X_3\{\geq 3\}X_7\ \{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
$\vee\ X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\}\ X_8\{\geq 3\}$
$\vee\ X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 3\}\ X_8\{< 2\}$
$\vee\ X_1\{\geq 2\}X_2\{< 2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{2\}X_3\{2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\}\ X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\}\ X_7\{2\}\ X_8\{2\}$
$\vee\ X_1\{\geq 2\}X_3\{\geq 3\}\ X_4\{\geq 2\}X_5\{< 3\}\ X_6\{\geq 2\}\ X_7\{2\}\ X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$

∨ $X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\}X_8\{2\}(0)$. (39)

Equation (39) is now simplified to yield the following equation (40), in which the remaining terms are mutually disjoint, and hence it is the desired PRE formula, which transforms, on a one-to-one basis into an expectation formula or network reliability by replacing each logic symbol by its expectation, and replacing the OR and AND operation by arithmetic addition and multiplication. Table 2 compares our initial expression (17) and final expression (40). The 8 terms in (17) have been replaced by 1+1+1+1+1+2+ 3+6 = 16 terms. In a sense, expression (17) remains 'shellable' up to its fifth term, while the sixth term was split into two terms, and the last two terms were replaced by three and six terms, respectively. The final multiplying factors introduced gradually via (12) and adjusted via (16) are distinguished in bold red in the right column of Table 2. What remains in black in this column is the variable instances that remained intact within a term.

Table 3 presents the gradual evolution of the lower and upper bounds of the expectation of system success, where we take the expectation of the sum of terms up to the highlighted one as a lower bound, and take the expectation of the sum of all terms as an upper bound. The two bounds finally converge to the same value, which agrees with (but is substantially more precise than) the result of 0.981902 reported in [1]. Correctness of the symbolic reliability corresponding to formula (40) might be validated via the tests in Rushdi [43].

$S = X_3\{\geq 3\}X_7\{\geq 3\}$

∨ $X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$

∨ $X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$

∨ $X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$

∨ $X_2\{2\}X_3\{2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}X_6\{\geq 2\} X_7\{\geq 3\}$

∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\} X_8\{< 2\}$

∨ $X_1\{\geq 2\}X_2\{< 2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{2\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$

∨ $X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{< 3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_3\{\geq 3\} X_4\{\geq 2\}X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$

∨ $X_1\{\geq 2\}X_2\{2\}X_3\{< 2\} X_4\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} X_4\{\geq 2\}X_5\{< 3\}X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$

∨ $X_1\{\geq 2\}X_2\{\geq 3\}X_3\{< 2\} X_4\{\geq 2\}X_5\{\geq 3\}X_6\{\geq 2\} X_7\{2\} X_8\{2\}$. (40)

Table 2. Comparison of the initial success expression (17) with the final success expression (40)

Initial success expression (17)	Final success expression (40)
$X_3\{\geq 3\}X_7\{\geq 3\}$	$X_3\{\geq 3\}X_7\{\geq 3\}$
∨ $X_2\{\geq 3\}X_7\{\geq 3\}$	∨ $X_2\{\geq 3\}X_3\{< 3\}X_7\{\geq 3\}$
∨ $X_3\{\geq 3\}X_5\{\geq 3\} X_8\{\geq 3\}$	∨ $X_3\{\geq 3\}X_5\{\geq 3\}X_7\{< 3\} X_8\{\geq 3\}$
∨ $X_2\{\geq 3\}X_5\{\geq 3\} X_8\{\geq 3\}$	∨ $X_2\{\geq 3\}X_3\{< 3\}X_5\{\geq 3\}$ $X_7\{< 3\} X_8\{\geq 3\}$
∨ $X_2\{\geq 2\}X_3\{\geq 2\} X_4\{\geq 2\}X_7\{\geq 3\} X_8\{\geq 2\}$	∨ $X_2\{2\}X_3\{2\} X_4\{\geq 2\}X_7\{\geq 3\}$ $X_8\{\geq 2\}$
∨ $X_1\{\geq 2\}X_2\{\geq 2\} X_3\{\geq 2\}X_6\{\geq 2\} X_7\{\geq 3\}$	∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{< 2\}$ $X_6\{\geq 2\} X_7\{\geq 3\}$ ∨ $X_1\{\geq 2\}X_2\{2\}X_3\{2\}X_4\{\geq 2\}X_6\{\geq 2\}$ $X_7\{\geq 3\} X_8\{< 2\}$

Initial success expression (17)	Final success expression (40)
$\vee\, X_1\{\geq 2\}X_3\{\geq 2\}\ X_4\{\geq 2\}\ X_6\{\geq 2\}$ $X_7\{\geq 2\}\ X_8\{\geq 2\}$	$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{< 2\}X_3\{2\}}\ X_4\{\geq 2\}$ $X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{2\}X_3\{2\}}\ X_4\{\geq 2\}$ $X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}X_3\{2\}}\ X_4\{\geq 2\}$ $\boldsymbol{X_5\{< 3\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$
$\vee\, X_1\{\geq 2\}X_2\{\geq 2\}\ X_4\{\geq 2\}X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$	$\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}X_3\{2\}}X_4\{\geq 2\}\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}\ X_8\{2\}}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}\boldsymbol{X_5\{< 3\}}$ $X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_3\{\geq 3\}}\ X_4\{\geq 2\}$ $\boldsymbol{X_5\{\geq 3\}}\ X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}\ X_8\{2\}}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{2\}X_3\{< 2\}}\ X_4\{\geq 2\}$ $X_6\{\geq 2\}\ X_7\{\geq 2\}\ X_8\{\geq 2\}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}X_3\{< 2\}}\ X_4\{\geq 2\}$ $\boldsymbol{X_5\{< 3\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}}\ X_8\{\geq 2\}$ $\vee\, X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}X_3\{< 2\}}\ X_4\{\geq 2\}$ $\boldsymbol{X_5\{\geq 3\}}X_6\{\geq 2\}\ \boldsymbol{X_7\{2\}\ X_8\{2\}}.$

Table 3. Gradual evolution of the lower and upper bounds of the expectation of system success. The two bounds finally converge to the same value, which agrees with (but is substantially more precise than) the result of 0.981902 reported in [1]

Equation number	Lower Bound	Upper Bound
19	0.800020	1.34 (reset to 1.0)
21	0.874930	1.140 (reset to 1.0)
23	0.9608161	0.9992377
25	0.968858	0.994590
27	0.971437	0.9880448
28	0.97179672	0.9880448
29	0.971875865	0.9880448
30	0.97299423	0.9880448
32	0.973144799	0.987894275
34	0.973323262	0.9877158119657848
35	0.9734248388	0.9877158119657848
38	0.97719701872	0.98397408250
40	0.981902222431299	0.981902222431299

6 Conclusions

We successfully imported the concept of a 'probability-ready expression' (PRE) from the two-valued logical domain to the multi-valued logical domain. This concept is of paramount importance since it allows a direct transformation of a random logical expression, on a one-to-one one, to its statistical expectation form, simply by replacing all logic variables by their statistical expectations, and also substituting arithmetic multiplication and addition for their logical counterparts (ANDing and ORing). We presented a general method for creating a symbolic multi-state PRE. The proposed method is illustrated with a detailed symbolic example of a real-case study of a multi-state delivery network. Our method produced a more precise version of the same numerical value that was obtained earlier by an alternative means.

The running example used for demonstrating PRE derivation in this paper is admittedly long and tedious. However, this example is very valuable from the pedagogical point of view, since it reveals many subtle points in the analysis. We are currently investigating other alternative techniques for obtaining the required multi-state reliability or success expectation. Use of the Inclusion-Exclusion (IE) Principle *per se* is out of question due to its exponential complexity. However, a factored IE method might prove fruitful when used in conjunction with a

limited-scope PRE derivation. A promising alternative is to attain multi-state PRE derivation through the use of the Boole-Shannon Expansion, which is essentially the essence of the celebrated RDSP method.

Competing Interests

Authors have declared that no competing interests exist.

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