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Average Life Prediction Based on Incomplete Data*

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Abstract

The two-parameter exponential distribution can often be used to describe the lifetime of products for example, electronic components, engines and so on. This paper considers a prediction problem arising in the life test of key parts in high speed trains. Employing the Bayes method, a joint prior is used to describe the variability of the parameters but the form of the prior is not specified and only several moment conditions are assumed. Under the condition that the observed samples are randomly right censored, we define a statistic to predict a set of future samples which describes the average life of the second-round samples, firstly, under the condition that the censoring distribution is known and secondly, that it is unknown. For several different priors and life data sets, we demonstrate the coverage frequencies of the proposed prediction intervals as the sample size of the observed and the censoring proportion change. The numerical results show that the prediction intervals are efficient and applicable.

Keywords: Prediction Interval, Incomplete Data, Bayes Method, Two-Parameter Exponential Distribution

1. Introduction

Prediction problem has been very often and useful in many fields of applications. The general prediction problem can be regarded as that of using the results of previous data to infer the results of future data from the same population. The lifetime of the second round sample is an important index in life testing experiments and in many situations people want to forecast the lifetimes of these samples as well as the system composed of these samples (See [1,2] and among others). For more details on the history of statistical prediction, analysis and application, see [3,4].

As we know, many quality characteristics are not normally distributed, especially the lifetime of products for example, electronic components, engines and so on. Assume that the lifetime of a component follows the two-parameter exponential distribution whose probability density function (pdf) given by

$$f(x;\theta,\mu)=\frac{1}{\theta}\exp\left(-\frac{x-\mu}{\theta}\right)I(x>\mu), \tag{1.1}$$

where $\theta>0$ and $\mu\ge 0$ are called the scale parameter and the location parameter, respectively, and $I(A)$ denotes the indicator function of the set A. The readers are referred to [5,6] for some practical applications of the two-parameter exponential distribution in real life. The recent relevant studies on the two-parameter exponential distribution can be found in [7-9], etc.

In this paper, we adopt the following testing scheme: for n groups of components, which come from n different manufacture units possessing the same technology and regulations, we sample m components from each group and put them to use at time $t=0$ and to practice economy the experiment will be terminated if one of the m components is ineffective, where m is a predetermined integer. Denote the lifetime of the ineffective component by $X_i\,(1\le i\le n)$. Obviously, $X_i=\min\{X_{i1},X_{i2},\cdots,X_{im}\}$, where $X_{ij}\,(1\le j\le m)$ is the life of the j-th component of the i-th group. Hence, we obtain n lifetime data $X_1,X_2,\cdots,X_n$. If $X_i\ge a$, where $a>0$ is a known constant, then we again sample one component from the i-th group and denote its unknown lifetime by Y_i. In this paper, our interest is to predict the average life of the second round sample, i.e., $\left[\sum_{i=1}^n I(X_i\ge a)\right]^{-1}\sum_{i=1}^n I(X_i\ge a)Y_i$. For instance, $k\left[\sum_{i=1}^n I(X_i\ge a)\right]^{-1}\sum_{i=1}^n I(X_i\ge a)Y_i$ approximately describes the average lifetime of a system of k components, based on the samples of the second round, is connected in active-parallel which fails only when all k components fail.

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Normally, there are two different views on prediction problems, the frequentist approach and the Bayes approach. The Bayesian viewpoint has received large attention for analyzing data in past several decades and has been often proposed as a valid alternative to traditional statistical perspectives (see [10-12], etc.). A main different point between the Bayes approach and the frequentist approach is that in Bayesian analysis we use not only the sample information but also the prior information of the parameter.

To adopt the Bayes approach, we regard the parameters θ and μ as the realization value of a random variable pair (Θ, U) with a joint prior distribution $G(\theta,\mu)$.

Let $(\theta_1,\mu_1),(\theta_2,\mu_2),\cdots,(\theta_n,\mu_n)$ be independent and identically distributed (i.i.d.) with the prior distribution $G(\theta,\mu)$, and conditionally on (θ_i,μ_i) assume X_{ij} has the pdf (1.1) which will be denoted by $f(x|\theta,\mu)$ in the following section. Set

$$S=\frac{\sum_{i=1}^{n} I(X_i \ge a)Y_i}{\sum_{i=1}^{n} I(X_i \ge a)}. \tag{1.2}$$

Our problem is how to construct a function $g(X_1,X_2,\cdots,X_n)$ to predict S.

As we know, many statistical experiments result in incomplete sample, even under well-controlled situations. This is because individuals will experience some other competing events which cause them to be removed. In life testing experiments, the experimenter may not always be in a position to observe the lifetimes of all components put on test due to time limitation or other restrictions(such as money, material resources, etc.) on data collection (see [9,13] and others). Hence, censored samples may arise in practice. In this paper we assume furthermore that $X_{ij}\,(1\le i\le n,\ 1\le j\le m)$ are censored from the right by nonnegative independent random variables $V_i\,(1\le i\le n)$ with a distribution function W. It is assumed that $X_{ij}\,(1\le i\le n,\ 1\le j\le m)$ are independent of $V_i\,(1\le i\le n)$. In the random censorship model, the true lifetimes $X_i=\min\{X_{i1},X_{i2},\cdots,X_{im}\}\,(1\le i\le n)$ are not always observable. Instead, we observe only $Z_i=\min\{X_i,V_i\}$ and $\delta_i=I(X_i\le V_i)$.

The paper is organized as follows. In Section 2, based on $(Z_i,\delta_i)(1\le i\le n)$, we define a predictive statistic for S and simulate its prediction results under the condition that the censoring distribution W is known. In Section 3, when the censoring distribution W is unknown, we obtain a similar result for a corresponding predictive statistic of S as well as demonstrate the prediction performances. Some conclusions and remarks are presented in Section 4 and Section 5 deals with the proofs of the main theorems.

2. Predictive Statistic for S with Known W

Note that X_{ij} has the conditional pdf $f(x|\theta,\mu)$, we know that, given (θ,μ), $X_i=\min\{X_{i1},X_{i2},\cdots,X_{im}\}$ has the pdf

$$l(x|\theta,\mu)=\frac{m}{\theta}\exp\left(-\frac{m(x-\mu)}{\theta}\right)I(x>\mu). \tag{2.1}$$

Since X_i and Y (if $X_i\ge a_i$) come from the same group, $(X_i,Y_i)(1\le i\le n)$ would be i.i.d. with common marginal pdf

$$p(x,y)=\int_{\bar{\Theta}\times\bar{U}} l(x|\theta,\mu)\,f(y|\theta,\mu)\,dG(\theta,\mu), \tag{2.2}$$

where $\bar{\Theta}\times\bar{U}$ denotes the support of the prior distribution $G(\theta,\mu)$.

Rewrite

$$S=\frac{\sum_{i=1}^{n} I(X_i\ge a)Y_i}{\sum_{i=1}^{n} I(X_i\ge a)}=\frac{n^{-1}\sum_{i=1}^{n} I(X_i\ge a)Y_i}{n^{-1}\sum_{i=1}^{n} I(X_i\ge a)}\hat{=}\frac{S_1}{S_2}. \tag{2.3}$$

By Fubini's theorem, we know Equation (2.4) (Below) and

$$\begin{aligned}ES_2&=E\left[n^{-1}\sum_{i=1}^{n} I(X_i\ge a)\right]=E\left\{E\left[I(X_i\ge a)\big|(\theta,\mu)\right]\right\}\\&=E_{(\theta,\mu)}\left[\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right],\end{aligned} \tag{2.5}$$

where $E_{(\theta,\mu)}$ denotes the expectation with respect to (θ,μ).

Based on $(Z_i,\delta_i)(1\le i\le n)$, we define

$$\begin{aligned}\bar{S}_1&=\frac{1}{n(n-1)}\sum_{i=1}^{n}\left[\sum_{j\ne i}^{n}\frac{Z_j\delta_j}{1-W(Z_j)}\right]\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)}\\&+(m-1)\frac{1}{n}\sum_{i=1}^{n}(Z_i-a)\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)},\end{aligned} \tag{2.6}$$

$$\begin{aligned}ES_1&=E\left[I(X_1\ge a)Y\right]_1=\int_0^\infty\int_0^\infty I(x\ge a)\,yp(x,y)\,dxdy=\int_{\bar{\Theta}\times\bar{U}}\left[\int_0^\infty I(x\ge a)\,l(x|\theta,\mu)\,dx\int_0^\infty yf(y|\theta,\mu)\,dy\right]dG(\theta,\mu)\\&=\int_{\bar{\Theta}\times\bar{U}}(\mu+\theta)\exp\left(-\frac{m(a-\mu)}{\theta}\right)dG(\theta,\mu)=E_{(\theta,\mu)}\left[(\mu+\theta)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right]\end{aligned} \tag{2.4}$$

and

$$\bar{S}_2 = \frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-W(Z_i)}. \quad (2.7)$$

Note that conditionally on (θ_i, μ_i) all $Z_i\,(1 \le i \le n)$ are i.i.d. with the distribution function

$$H = 1-[1-W(z)][1-L(z|\theta,\mu)],$$

where $L(z|\theta,\mu) = \int_0^z l(x|\theta,\mu)dx,$

we have Equation (2.8) (Below)

and

$$\begin{aligned} E\bar{S}_2 &= E\left[\frac{I(Z_1 \ge a)\delta_1}{1-W(Z_1)}\right] \\ &= E_{(\theta,\mu)}\left\{\iint_{x\le v}\frac{I(x\ge a)}{1-W(x)}dW(v)l(x|\theta,\mu)dx\right\} \quad (2.9) \\ &= E_{(\theta,\mu)}\left[\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] = ES_2. \end{aligned}$$

Hence, the statistics $\bar{S}_i\,(i=1,2)$ have the same expectation as $S_i\,(i=1,2)$.

Set

$$\bar{S} = \frac{\bar{S}_1}{\bar{S}_2}. \quad (2.10)$$

Remark 1. Note that it is almost impossible that all δ_i's are equal to zero, so $\bar{S}_1$ and $\bar{S}_2$ are reasonable estimators for S_1 and S_2, respectively.

The main result in this section can be formulated in the following theorem.

Theorem 1. If the following conditions are satisfied:

1) $E\mu^2 < \infty,\ E\theta^2 < \infty;$

2) $E[\phi(\theta,\mu)\psi(\theta,\mu)] < \infty;$

3) $E\left[\frac{X^2 I(X \ge a)}{(1-W(X))^2}\right] < \infty;$

then

$$\bar{S} - S \xrightarrow{p} 0,\ n \to \infty$$

where $\phi(\theta,\mu) = E\left[\frac{X^2}{1-W(X)}\middle|(\theta,\mu)\right],$

$\psi(\theta,\mu) = E\left[\frac{I(X \ge a)}{1-W(X)}\middle|(\theta,\mu)\right]$ and $\xrightarrow{p}$ denotes convergence in probability.

Clearly, $\bar{S}$ can be used as a predictive statistic for S in this case.

Especially, when there is no censorship $(Z_i = X_i,\ \delta_i = 1)$, $\bar{S}_1$ and $\bar{S}_2$ turn into, respectively

$$\begin{aligned} \bar{S}_{10} &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\left[\sum_{j\ne i}^{n}X_j\right]I(X_i \ge a) \\ &+ (m-1)\frac{1}{n}\sum_{i=1}^{n}(X_i - a)I(X_i \ge a) \end{aligned} \quad (2.11)$$

and

$$\bar{S}_{20} = \frac{1}{n}\sum_{i=1}^{n}I(X_i \ge a). \quad (2.12)$$

Consequently, we use $\bar{S}_0 = \bar{S}_{10}/\bar{S}_{20}$ as a prediction statistics of S.

Normally, we choose Gamma prior for the parameters (θ,μ), however, it is easy to see that Theorem 1 does not depend on any specific prior distribution. This shows that for any a prior distribution satisfying the conditions of Theorem 1, the conclusion of Theorem 1 will hold.

So in the simulation study, we let the prior distribution of parameters

$$\mu \sim Uniform(500,1300), \quad (2.13)$$

$$\theta \sim Uniform(800,1400), \quad (2.14)$$

and the censoring distribution be

$$\begin{aligned} E\bar{S}_1 &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\ne i}^{n}E\left\{E\left[\frac{Z_j\delta_j}{1-W(Z_j)}\middle|(\theta,\mu)\right]E\left[\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)}\middle|(\theta,\mu)\right]\right\} \\ &+ (m-1)E\left\{E\left[\frac{(Z_1-a)I(Z_1\ge a)\delta_1}{1-W(Z_1)}\middle|(\theta,\mu)\right]\right\} \\ &= E\left\{\iint_{x\le v}\frac{x}{1-W(x)}dW(v)l(x|\theta,\mu)dx\iint_{x\le v}\frac{I(x\ge a)}{1-W(x)}dW(v)l(x|\theta,\mu)dx\right\} \\ &+ (m-1)E\left\{\iint_{x\le v}\frac{(x-a)I(x\ge a)}{1-W(x)}dW(v)l(x|\theta,\mu)dx\right\} \\ &= E_{(\theta,\mu)}\left[\left(\mu+\frac{\theta}{m}\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)+\frac{(m-1)\theta}{m}\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] = ES_1, \end{aligned} \quad (2.8)$$

$$W(v) = 1 - \exp(-cv),\ v > 0, \tag{2.15}$$

where $c = 0.0001, 0.0002$ and 0.00025 can be used to describe the censorship proportion (CP) $P(X > V)$, which denotes the probability that X is larger than V. In the censorship model, if the probability $P(X > V)$ gets larger, then more V_i's are likely to be observed other than X_i's. Also, let $m = 3$.

Under the above assumptions, it is not difficult to check that the conditions (i), (ii) and (iii), defined in Theorem 1, are satisfied. Note that $EX = E\mu + E\theta/m = 3800/3$, which shows the mean time to failure (MTTF) of minimum lifespan of the m components is 3800/3.

Firstly, we generate n random values from the priors (2.13) and (2.14), and denote them by $(\theta_i, \mu_i)(1 \le i \le n)$. Secondly, by Equations (2.1) and (2.15) we obtain $X_i (1 \le i \le n)$ and $V_i (1 \le i \le n)$, accordingly, we get $Z_i = \min\{X_i, V\}_i (1 \le i \le n)$ and $\delta_i = I(X_i \le V_i)(1 \le i \le n)$. Thirdly, let the predetermined constant a be equal to the MTTF, we compute the frequencies of the event $\{|\bar{S} - S| < \varepsilon\}$ for $\varepsilon = 200$ and $\varepsilon = 100$ with $c = 0.0001, 0.0002, 0.00025$. Repeating the process for 5000 times, the results are reported in **Table 1**.

where PCP denotes the practical CP obtained from the simulation data. From **Table 1**, firstly, we find that when the PCP is fixed, the frequencies of $\{|\bar{S} - S| < \varepsilon\}$ generally increase as the sample size n and ε get larger, respectively. Secondly, as it can be expected, the frequencies tend to decrease as the PCP increases. As a contrast, we report the frequencies of $\{|\bar{S}_0 - S| < \varepsilon\}$ in **Table 2** when there is no censorship, which are uniformly better than those of **Table 1**.

In **Tables 3** and **4**, we change the value of the constant a and present the frequencies of $\{|\bar{S} - S| < \varepsilon\}$ with a = 0.5MTTF and a = 1.5MTTF, respectively.

Although it is difficult to describe how S and the frequencies depend on the constant a, there is a trend that the frequencies of $\{|\bar{S} - S| < \varepsilon\}$ are getting larger as the constant a becomes smaller. The reason is that S denotes the average values of Y_i's, as we know, if a gets smaller, obviously, both $\sum_{i=1}^{n} I(X_i \ge a) Y_i$ and $\sum_{i=1}^{n} I(X_i \ge a)$ will become larger, then the S is more like the average life other than denoting several or a few samples, hence, the proposed method works better. That is why the frequencies in **Table 3** perform the best among the above tables, especially for larger n.

It is well-known that the prior distribution $G(\theta, \mu)$ reflects the past experience about the parameter (θ, μ) in Bayesian analysis. During the process of our simulation, we find the fact that the performance of the frequencies of $\{|\bar{S} - S| < \varepsilon\}$ depends on the prior distribution. In what follows, we generate three kinds data of different prior and report the corresponding performances of the frequencies in **Figure 1** and **Table 5**. At the same time, we simulate the performance of the frequencies of $\{|\bar{S}_0 - S|\}$ as a contrast.

Table 1. *a* = MTTF = 3800/3.

c	n	Frequency $\{\lvert\bar{S} - S\rvert < \varepsilon\}$	
		$\varepsilon = 200$	$\varepsilon = 100$
0.0001 (PCP = 11.80%)	20	0.7880	0.6020
	30	0.8500	0.7040
	50	0.9300	0.7900
0.0002 (PCP = 22.10%)	20	0.7100	0.5360
	30	0.7780	0.5980
	50	0.8680	0.7220
0.00025 (PCP = 26.80%)	20	0.5800	0.4660
	30	0.6720	0.5340
	50	0.8040	0.6320

Table 2. *a* = MTTF = 3800/3.

n	Frequency $\{\lvert\bar{S}_0 - S\rvert < \varepsilon\}$	
	$\varepsilon = 200$	$\varepsilon = 100$
20	0.8280	0.6820
30	0.8610	0.7640
50	0.9440	0.8010

Table 3. *a* = 0.5 MTTF.

c	n	Frequency $\{\lvert\bar{S} - S\rvert < \varepsilon\}$	
		$\varepsilon = 200$	$\varepsilon = 100$
0.0001 (PCP = 11.80%)	20	0.8080	0.6620
	30	0.8800	0.8140
	50	0.9350	0.8410
0.0002 (PCP = 22.10%)	20	0.7300	0.6390
	30	0.8470	0.6980
	50	0.8980	0.8320
0.00025 (PCP = 26.80%)	20	0.6300	0.5670
	30	0.7440	0.6340
	50	0.8620	0.7920

Table 4. a = 1.5 MTTF.

c	n	Frequency $\{\lvert\bar{S}-S\rvert<\varepsilon\}$	
		$\varepsilon = 200$	$\varepsilon = 100$
0.0001 (PCP = 11.80%)	20	0.6860	0.5920
	30	0.7600	0.6060
	50	0.9050	0.6800
0.0002 (PCP = 22.10%)	20	0.6100	0.5160
	30	0.6760	0.5770
	50	0.7690	0.6240
0.00025 (PCP = 26.80%)	20	0.4880	0.4260
	30	0.5710	0.5130
	50	0.7050	0.4650

From the above, we see that for the three different priors of (θ,μ), which have the same prior means for θ and μ, respectively, under the condition that their PCPs are almost the same, the more concentrative the prior values, the better the performances of the frequencies. This numerical evidence means that the proposed prediction intervals are in accordance with practice and applicable.

3. Predictive Statistic for S with Unknown W

Note that the censoring distribution $W(.)$ is unknown, hence the predictive statistic $\bar{S}$ is unavailable to use. This leads us to adopt the product limit estimator, which introduced to statistical problems by [14], to propose a corresponding predictive statistic for S in this case.

Define

$$\hat{S}=\frac{\hat{S}_1}{\hat{S}_2}, \tag{3.1}$$

where

$$\begin{aligned}\hat{S}_1=&\frac{1}{n(n-1)}\sum_{i=1}^{n}\left[\sum_{j\neq i}^{n}\frac{Z_j\delta_j}{1-\hat{W}_n(Z_j)}\right]\frac{I(Z_i\geq a)\delta_i}{1-\hat{W}_n(Z_i)}\\&+(m-1)\frac{1}{n}\sum_{i=1}^{n}(Z_i-a)\frac{I(Z_i\geq a)\delta_i}{1-\hat{W}_n(Z_i)},\end{aligned} \tag{3.2}$$

$$\hat{S}_2=\frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i\geq a)\delta_i}{1-\hat{W}_n(Z_i)}, \tag{3.3}$$

and the product limit estimator $\hat{W}_n(t)$ is given by

$$1-\hat{W}_n(t)=\prod_{i=1}^{n}\left[\frac{n-i}{n-i+1}\right]^{I(Z_{(i)}\leq t,\delta_{(i)}=0)},\ t<Z_{(n)}, \tag{3.4}$$

where $Z_{(1)}\leq Z_{(2)}\leq\cdots\leq Z_{(n)}$ are the order statistics of $(Z_1,Z_2,\cdots,Z_n)$ and $\delta_{(i)}$ is the concomitant of $Z_{(i)}$.

Theorem 2. Under the same conditions as Theorem 1, we have

$$\hat{S}-S\xrightarrow{p}0.$$

Obviously, $\hat{S}$ can be used as a predictive statistic for S when the censoring distribution W is unknown.

Remark 2. Note that in the case that the censoring distribution W is unknown, it is impossible to check whether the conditions of Theorem 2 are satisfied or not. Hence, we first need to propose a distribution function W to fit the data $V_1,V_2,\cdots,V_n$.

Consider the following several data sets, which come from [15]. We take them as the the censoring variables $V_1,V_2,\cdots,V_n$.

$n=5$, 381, 395, 408, 423, 431.

$n=20$, 350, 380, 400, 430, 450, 470, 480, 500, 520, 540, 550, 570, 600, 610, 630, 650, 670, 730, 770, 840.

$n=31$, 30 926, 34 554, 36 381, 38 423, 40 103, 40 501, 42 200, 44 392, 46 092, 46 125, 46 175, 48 025, 48 025, 48 055, 48 055, 48 055, 48 055, 48 056, 51 675, 52 344, 52 345, 52 345, 52 345, 52 379, 55 997, 56 202, 57 709, 57 709, 57 709, 57 709, 63 496.

$n=71$, 3 95642, 4 00418, 4 09161, 4 35505, 4 35540, 4 37601, 4 39179, 4 48768, 4 48768, 4 73667, 4 73667, 4 93985, 4 96362, 5 22019, 5 35341, 5 37272, 5 418045 44411, 5 60317, 5 69810, 5 74617, 5 84352, 6 17514, 6 19741, 6 24969, 6 27976, 6 27976, 6 57274, 6 74048, 6 88765, 7 18309, 7 20900, 7 20900, 7 20900, 7 20900, 7 36640, 7 58164, 7 58164, 7 58164, 7 64559, 8 24600, 8 5997, 8 71397, 8 93634, 9 04422, 9 197 45, 9 51173, 9 75447, 9 96745, 10 13631, 10 17288, 10 17288, 10 30804, 10 39500, 10 609 23, 10 60923, 10 78897, 10 87997, 10 97175, 11 59441, 11 99059, 12 23731, 12 40031, 12 40031, 12 55001, 13 19873, 13 94778, 15 55712, 17 646 12, 19 84823, 23 19907.

As we know the Weibull distribution is widely applied to life testing and reliability analysis. Some studies on it have been quickly developed in recent years (see [16] and [17], etc). The cumulative distribution function (CDF) of the three-parameter Weibull distribution is

$$F(t)=1-\exp\left(-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right), \tag{3.5}$$

where β is the shape parameter, γ is the location parameter and η is the scale parameter.

Employing the method proposed by [18], we use the three-parameter Weibull distribution to fit the above four groups data and test the fitting by Kolmogorov-Smirnov

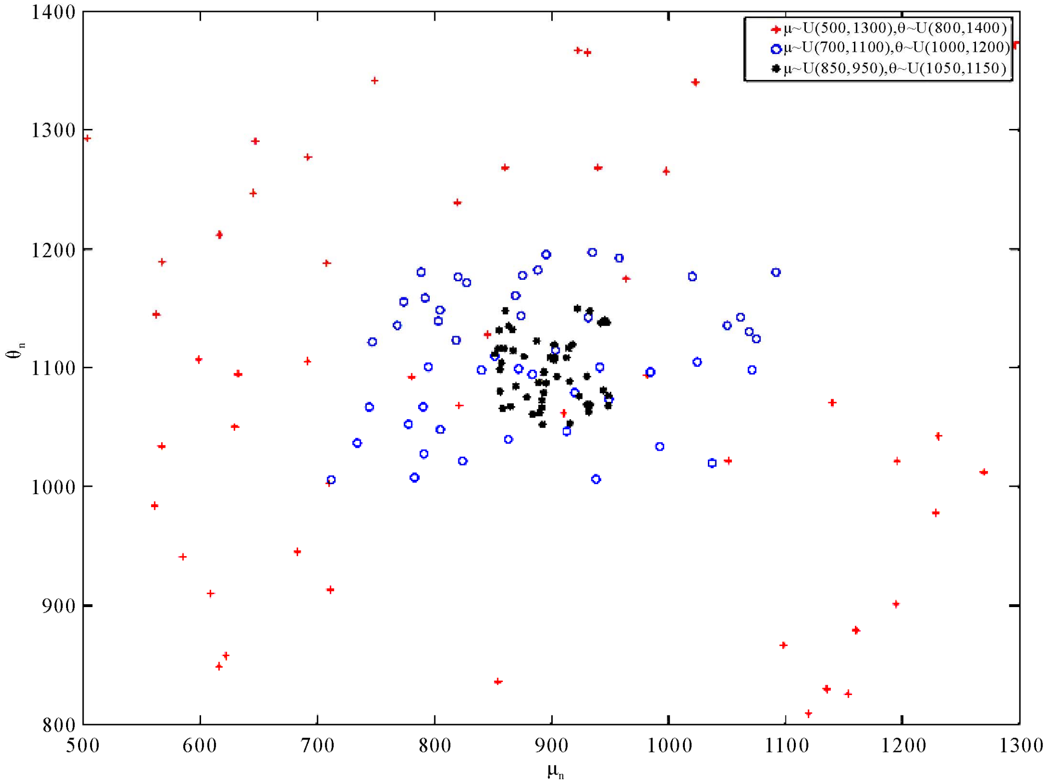

Figure 1. The values of (μ_n, θ_n) for three different priors.

Table 5. *a* = 0.5 MTTF.

PCP and the prior	n	Frequency $\{\lvert\bar{S}-S\rvert<\varepsilon\}$		Frequency $\{\lvert\bar{S}_0-S\rvert<\varepsilon\}$	
		$\varepsilon=200$	$\varepsilon=100$	$\varepsilon=200$	$\varepsilon=100$
PCP=11.76% $\mu\sim U(500,1300)$ $\theta\sim U(800,1400)$	50	0.9240	0.6730	0.9420	0.7060
PCP=11.94% $\mu\sim U(700,1100)$ $\theta\sim U(1000,1200)$	50	0.9300	0.7280	0.9480	0.7520
PCP=11.91% $\mu\sim U(850,950)$ $\theta\sim U(1050,1150)$	50	0.9370	0.7540	0.9510	0.8500

s (K-S) test method. Note that (3.5) can be transformed into the following linear equation

$$\ln\left(-\ln\left[1-F(t)\right]\right)=\beta\ln(t-\gamma)-\beta\ln(\eta). \quad (3.6)$$

Firstly, to estimate the location parameter γ, we follow the principle of golden-section to find a value which located in the interval $(0, t_{(1)})$ to maximize the absolute value of the correlation coefficient defined by

$$\rho=\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}},$$

where $\bar{x} = \sum_{i=1}^{n} x_i / n$, $\bar{y} = \sum_{i=1}^{n} y_i / n$, $x_i = \ln\left(t_{(i)} - \gamma\right)$, $y_i = \ln\left(-\ln\left[1 - F\left(t_{(i)}\right)\right]\right)$ and $t_{(1)}, t_{(2)}, \cdots, t_{(n)}$ are the statistics of $t_1, t_2, \cdots, t_n$ and $F\left(t_{(i)}\right) = (i - 0.32)/(n + 0.36)$.

Secondly, adopting the least square method and regarding every group data as $t_1, t_2, \cdots, t_n$, we report the fitting results and the estimators of the parameters as well as the Kolmogorov-Smirnov test values in **Figures 2-5** and **Table 6**. Where the K-S value denotes the Kolmogorov-Smirnov test value.

Assume that the parameters (θ, μ) have priors similar to (2.13) and (2.14), for example $\mu \sim Uniform(\mu_1, \mu_2)$, $\theta \sim Uniform(\theta_1, \theta_2)$, and $m = 3$. Obviously, under the condition that the censoring distribution W is Weibull distribution with the above estimated parameters, it is easy to check that the conditions of Theorem 2 are satisfied.

We simulate the frequencies of $\{|\hat{S} - S| < \varepsilon\}$ as the PCP changes and present the results in what follows. Also, we refer to the performances of the frequencies of $\{|\bar{S}_0 - S| < \varepsilon\}$ as a contrast.

From **Tables 7-9**, firstly, it is the same as before, we find that for the fixed PCP, the frequencies of $\{|\hat{S} - S| < \varepsilon\}$ and $\{|\bar{S}_0 - S| < \varepsilon\}$ generally increase as the Constant a gets small. Secondly, compared with **Tables 1**, **3** and **4**, for the same sample size n, the frequencies of $\{|\bar{S} - S| < \varepsilon\}$ generally tend to be larger than those of $\{|\hat{S} - S| < \varepsilon\}$, which means in this case for given ε $\bar{S}$ is more concentrated in the vicinity of S. However, this may not be the case all the time. One reason is that X_i's and V_i's are different even for each the same sample size n and hence this makes the comparison more complicated. Thirdly, consistently, whether W is known or not the performances of the frequencies of $\{|\bar{S}_0 - S| < \varepsilon\}$ are the best. Also, as it can be expected, the frequencies of $\{|\hat{S} - S| < \varepsilon\}$ generally tend to decrease as the PCP increases.

4. Conclusions and Remarks

In this paper, assume the observed lifetimes of components are rightly censored, we define a prediction statistic to predict the average value of some untested components, firstly, under the condition that the censoring distribution is known and secondly, that it is unknown. In the case that the censoring distribution is unknown, we first fit the data $V_1, V_2, \cdots, V_n$ with a distribution function, say $W(t)$, and test the fitting by Kolmogorov-Smirnov

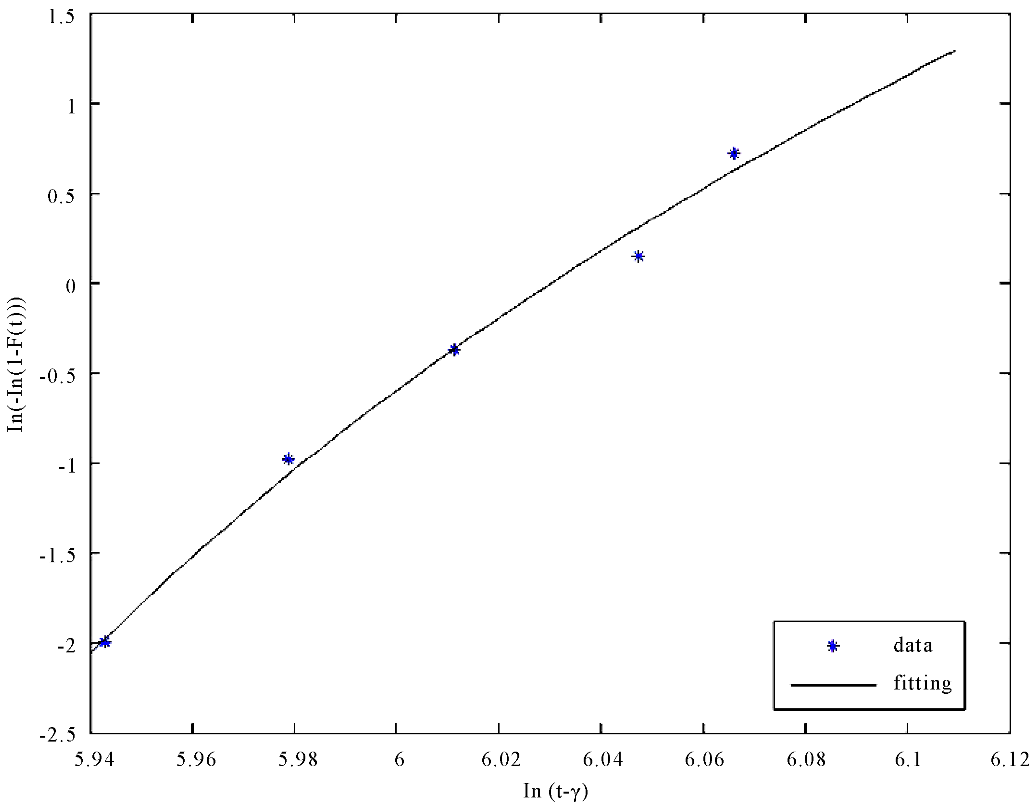

Figure 2. The case of n = 5.

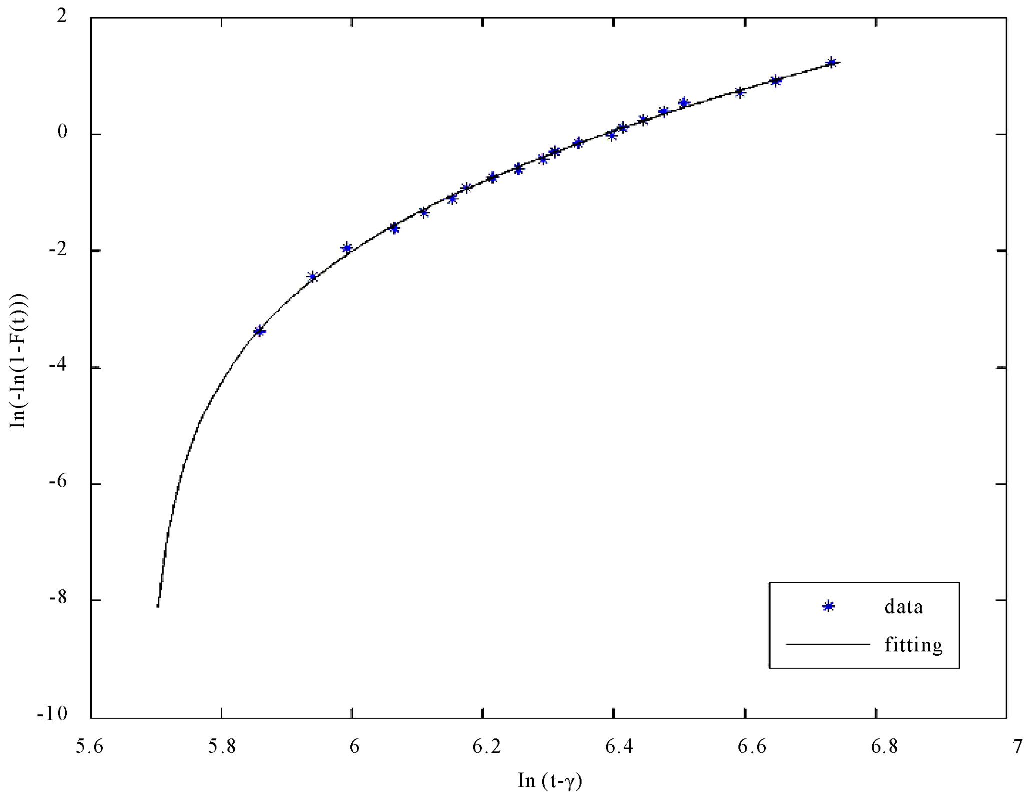

Figure 3. The case of n = 20.

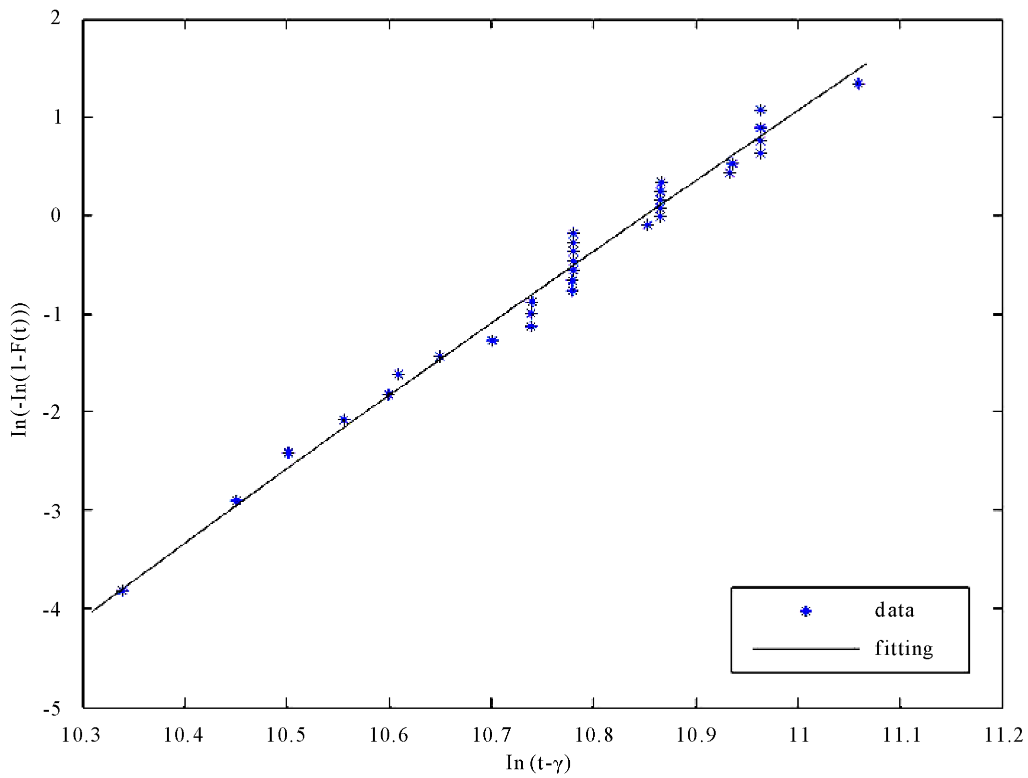

Figure 4. The case of n = 31.

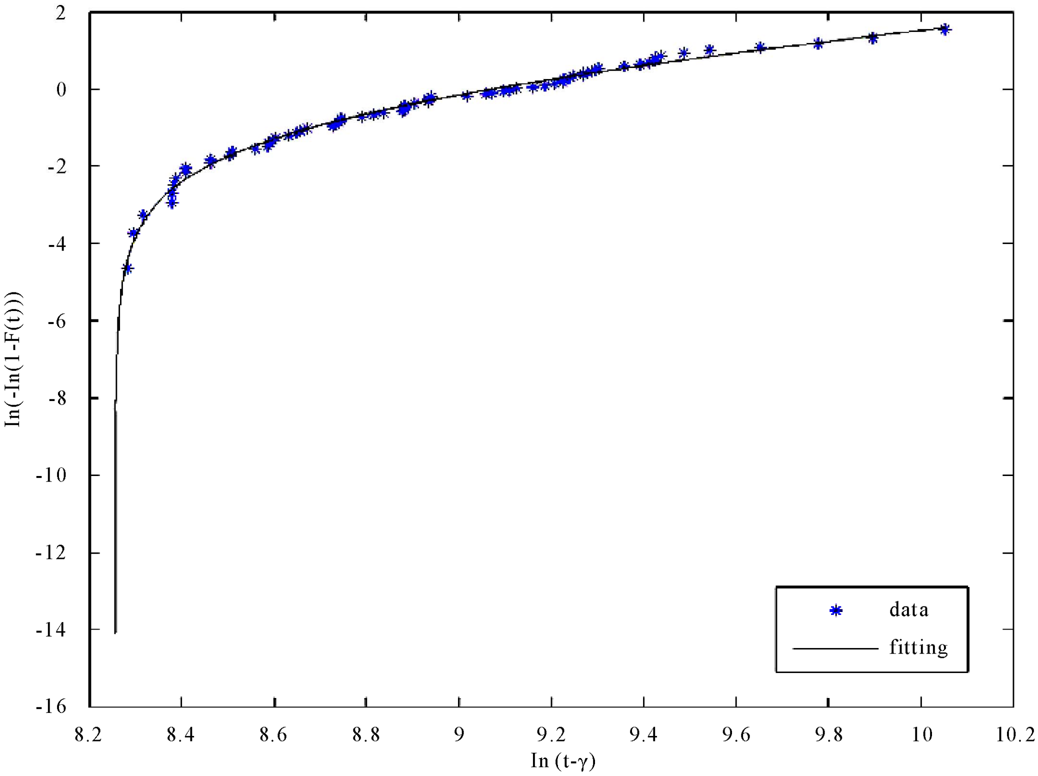

Figure 5. The case of n = 71.

Table 6. Parameter estimation and test.

	β	γ	η	correlation coefficient	K-S value
n = 5	3.9333	327.8554	87.9323	0.9552	0.0577
n = 20	1.9868	294.9828	298.2851	0.9992	0.0261
n = 31	6.5704	4623.1700	46952.0369	0.9914	0.1129
N = 71	1.1500	3854.0232	4856.6078	0.9960	0.0481

Table 7. $a = 1.5(E\mu + E\theta/m)$ **.**

		Frequency $\{\lvert\hat{S} - S\rvert < \varepsilon\}$		Frequency $\{\lvert\bar{S}_0 - S\rvert < \varepsilon\}$	
PCP	n	$\varepsilon = 200$	$\varepsilon = 100$	$\varepsilon = 200$	$\varepsilon = 100$
10.91%	5	0.5630	0.5330	0.7100	0.7510
4.33%	20	0.6600	0.6520	0.8160	0.7190
6.63%	31	0.7100	0.6800	0.8500	0.7630
10.08%	71	0.8770	0.7420	0.9300	0.7730

Table 8. $a = E\mu + E\theta/m$ **.**

		Frequency $\{\lvert\hat{S} - S\rvert < \varepsilon\}$		Frequency $\{\lvert\bar{S}_0 - S\rvert < \varepsilon\}$	
PCP	n	$\varepsilon = 200$	$\varepsilon = 100$	$\varepsilon = 200$	$\varepsilon = 100$
10.91%	5	0.5610	0.5500	0.7190	0.6710
4.33%	20	0.6740	0.6500	0.8260	0.6990
6.63%	31	0.7940	0.7120	0.8300	0.7530
10.08%	71	0.9020	0.7950	0.9380	0.8530

Table 9. $a = 0.5(E\mu + E\theta/m)$.

		Frequency $\{\lvert\hat{S} - S\rvert < \varepsilon\}$		Frequency $\{\lvert\bar{S}_0 - S\rvert < \varepsilon\}$	
PCP	n	$\varepsilon = 200$	$\varepsilon = 100$	$\varepsilon = 200$	$\varepsilon = 100$
10.91%	5	0.5930	0.5030	0.7390	0.6910
4.33%	20	0.6990	0.6170	0.8360	0.6890
6.63%	31	0.8030	0.7400	0.9230	0.7730
10.08%	71	0.9110	0.8100	0.9300	0.8380

test. Then, we regard the data $V_1, V_2, \cdots, V_n$ as being distributed according to $W(t)$ and check whether the conditions of Theorem 2 are satisfied or not. The numerical evidences show that the proposed prediction intervals are in accordance with practice and applicable. Also, it is easy to see that the proposed prediction method can be extended to many important survival models such as Erlang distribution, Gompertz distribution and so on. Furthermore, we may consider the same prediction problem in any a pdf, say $f(x,\theta)I(x>0)$, which may be a finite mixture of any two life distributions, which occurs when two different causes of failure are present (see [19] and among others).

5. Proofs

5.1. The Proof of Theorem 1

Proof. In order to obtain the conclusion of Theorem 1, we first prove

$$\bar{S}_1 - S_1 \xrightarrow{p} 0, \ n \to \infty. \tag{5.1}$$

Note that

$$E\left(\bar{S}_1 - S_1\right)^2 = ES_1^2 - 2ES_1\bar{S}_1 + E\bar{S}_1^2. \tag{5.2}$$

Firstly, it is easy to see Equation (5.3) (Below)
Secondly, we have belowing Equation (5.4)

$$\begin{aligned} ES_1^2 &= \frac{1}{n^2} E\left[\sum_{i=1}^{n} I\left(X_i \ge a\right) Y_i^2 + \sum_{i \ne j} I\left(X_i \ge a\right) I\left(X_j \ge a\right) Y_i Y_j\right] \\ &= \frac{1}{n} E\left[\left(\mu^2 + 2\theta^2 + 2\mu\theta\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] + \frac{n-1}{n} E\left[(\mu+\theta)^2 \exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right]. \end{aligned} \tag{5.3}$$

$$\begin{aligned} ES_1\bar{S}_1 &= \frac{1}{n^2(n-1)} \sum_{i=1}^{n} E\left[I\left(X_i \ge a\right) Y_i \left(\sum_{j \ne i}^{n} \frac{Z_j \delta_j}{1 - W\left(Z_j\right)}\right) \frac{I\left(Z_i \ge a\right)\delta_i}{1 - W\left(Z_i\right)}\right] \\ &+ \frac{1}{n^2(n-1)} \sum_{i \ne j} E\left[I\left(X_i \ge a\right) Y_i \left(\sum_{k \ne j}^{n} \frac{Z_k \delta_k}{1 - W\left(Z_k\right)}\right) \frac{I\left(Z_j \ge a\right)\delta_j}{1 - W\left(Z_j\right)}\right] \\ &+ \frac{m-1}{n^2} \sum_{i=1}^{n} E\left[\left(\left(Z_i - a\right)\frac{I\left(Z_i \ge a\right)\delta_i}{1 - W\left(Z_i\right)}\right) I\left(X_i \ge a\right) Y_i\right] + \frac{m-1}{n^2} \sum_{i \ne j}^{n} E\left[\left(\left(Z_i - a\right)\frac{I\left(Z_i \ge a\right)\delta_i}{1 - W\left(Z_i\right)}\right) I\left(X_j \ge a\right) Y_j\right] \\ &= \frac{1}{n} E\left[(\mu+\theta)\left(\mu + \frac{\theta}{m}\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] + \frac{1}{n} E\left[(\mu+\theta)\left(a + \frac{\theta}{m}\right)\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right] \\ &+ \frac{n-2}{n} E\left[(\mu+\theta)\left(\mu + \frac{\theta}{m}\right)\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right] + \frac{1}{n}\cdot\frac{m-1}{m} E\left[\theta(\mu+\theta)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] \\ &+ \frac{n-1}{n}\cdot\frac{m-1}{m} E\left[\theta(\mu+\theta)\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right]. \end{aligned} \tag{5.4}$$

Thirdly, $\bar{S}_1$ can be represented as

$$\bar{S}_1 = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{n-1}\sum_{j \ne i}^{n} \frac{Z_j \delta_j}{1 - W\left(Z_j\right)} + (m-1)\left(Z_i - a\right)\right] \frac{I\left(Z_i \ge a\right)\delta_i}{1 - W\left(Z_i\right)}. \tag{5.5}$$

We know

$$E\bar{S}_1^2 = \frac{1}{n^2}\left(\sum_{i=1}^{n} Q_i + \sum_{1\le i\ne j\le n} Q_{ij}\right), \tag{5.6}$$

where

$$\begin{aligned} Q_i &= E\left\{\left[\frac{1}{n-1}\sum_{j\ne i}^{n}\frac{Z_j\delta_j}{1-W(Z_j)} + (m-1)(Z_i-a)\right]^2\left[\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)}\right]^2\right\} \\ &= \frac{1}{n-1}E\left\{E\left[\frac{X^2}{1-W(X)}\middle|(\theta,\mu)\right]E\left[\frac{I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} + \frac{n-2}{n-1}E\left\{\left(\mu+\frac{\theta}{m}\right)^2 E\left[\frac{I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} \\ &\quad + (m-1)^2 E\left\{E\left[\frac{(X-a)^2 I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} + 2(m-1)E\left\{\left(\mu+\frac{\theta}{m}\right)E\left[\frac{(X-a)I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\}, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} Q_{ij} &= \frac{1}{(n-1)^2}E\left\{\left[\sum_{k\ne i}^{n}\frac{Z_k\delta_k}{1-W(Z_k)}\right]\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)}\left[\sum_{l\ne j}^{n}\frac{Z_l\delta_l}{1-W(Z_l)}\right]\frac{I(Z_j\ge a)\delta_j}{1-W(Z_j)}\right\} + \frac{m-1}{n-1}E\left\{\left[\sum_{k\ne i}^{n}\frac{Z_k\delta_k}{1-W(Z_k)}\right]\frac{I(Z_i\ge a)\delta_i}{1-W(Z_i)}\frac{(Z_j-a)I(Z_j\ge a)\delta_j}{1-W(Z_j)}\right\} \\ &\quad + \frac{m-1}{n-1}E\left\{\left[\sum_{l\ne j}^{n}\frac{Z_l\delta_l}{1-W(Z_l)}\right]\frac{I(Z_j\ge a)\delta_j}{1-W(Z_j)}\frac{(Z_i-a)I(Z_i\ge a)\delta_i}{1-W(Z_i)}\right\} + (m-1)^2 E\left\{\frac{(Z_i-a)I(Z_i\ge a)\delta_i}{1-W(Z_i)}\frac{(Z_j-a)I(Z_j\ge a)\delta_j}{1-W(Z_j)}\right\} \\ &= \frac{n-2}{(n-1)^2}E\left\{E\left[\frac{X^2}{1-W(X)}\middle|(\theta,\mu)\right]\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right\} + \frac{(n-2)(n-3)}{(n-1)^2}E\left\{\left(\mu+\frac{\theta}{m}\right)^2\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right\} \\ &\quad + \frac{2(n-2)}{(n-1)^2}E\left\{E\left[\frac{XI(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\left(\mu+\frac{\theta}{m}\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right\} + \frac{1}{(n-1)^2}E\left\{E^2\left[\frac{XI(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} \\ &\quad + \frac{2(m-1)}{n-1}E\left\{E\left[\frac{X(X-a)I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right\} \\ &\quad + \frac{2(n-2)}{n-1}\cdot\frac{m-1}{m}E\left[\theta\left(\mu+\frac{\theta}{m}\right)\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right] + \frac{(m-1)^2}{m^2}E\left[\theta^2\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right]. \end{aligned} \tag{5.8}$$

Along with Equations (5.3)-(5.4) and (5.6)-(5.8), we have

$$\begin{aligned} E\left(\bar{S}_1-S_1\right)^2 &= \frac{1}{n}E\left[\left(-\mu^2-2\theta\mu\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right] + \frac{-n+3}{n(n-1)}E\left[\left(\mu+\frac{\theta}{m}\right)^2\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right] \\ &\quad -\frac{2}{n}E\left[(\mu+\theta)\left(a+\frac{\theta}{m}\right)\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right] + \frac{1}{n(n-1)}E\left\{E\left[\frac{X^2}{1-W(X)}\middle|(\theta,\mu)\right]E\left[\frac{I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} \\ &\quad + \frac{n-2}{n(n-1)}E\left\{\left(\mu+\frac{\theta}{m}\right)^2 E\left[\frac{I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} + \frac{(m-1)^2}{n}E\left\{E\left[\frac{(X-a)^2 I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} \\ &\quad + \frac{2(m-1)}{n}E\left\{\left(\mu+\frac{\theta}{m}\right)E\left[\frac{(X-a)I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\} + \frac{n-2}{n(n-1)}E\left\{E\left[\frac{X^2}{1-W(X)}\middle|(\theta,\mu)\right]\exp\left(-\frac{2m(a-\mu)}{\theta}\right)\right\} \\ &\quad + \frac{2(n-2)}{n(n-1)}E\left\{E\left[\frac{XI(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\left(\mu+\frac{\theta}{m}\right)\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right\} \\ &\quad + \frac{2(m-1)}{n}E\left\{E\left[\frac{X(X-a)I(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\exp\left(-\frac{m(a-\mu)}{\theta}\right)\right\} + \frac{1}{n(n-1)}E\left\{E^2\left[\frac{XI(X\ge a)}{1-W(X)}\middle|(\theta,\mu)\right]\right\}. \end{aligned} \tag{5.9}$$

Combining Equation (5.9) and using the following facts:

1) $\phi(\theta,\mu) > E\left[X^2\middle|(\theta,\mu)\right] > \left(\mu+\frac{\theta}{m}\right)^2$;

2) $E\left[\frac{X^2}{1-W(X)}\right] \le \frac{a^2}{1-W(a)} + E\left[\frac{X^2 I(X \ge a)}{(1-W(X))^2}\right]$;

3)

$$E\left[\frac{X^2 I(X \ge a)}{(1-W(X))^2}\right] < \infty \Rightarrow E\left[\frac{(X-a)^2 I(X \ge a)}{(1-W(X))^2}\right] < \infty; \tag{5.10}$$

and also by Cauchy-Schwarz inequality, we easily know that under the conditions 1), 2) and 3),

$$\lim_{n\to\infty} E\left(\bar{S}_1 - S_1\right)^2 = 0. \tag{5.11}$$

Then by Markov's inequality, we conclude that (5.1) holds.

On the other hand, note that as $n \to \infty$

$$\bar{S}_2 - S_2 = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{I(Z_i \ge a)\delta_i}{1-W(Z_i)} - I(X_i \ge a)\right] \to 0, \tag{5.12}$$

with probability 1,

and

$$S_2 \to P(X \ge a) \text{ with probability } 1. \tag{5.13}$$

From Equations (5.1), (5.12) and (5.13), Theorem 1 follows.

5.2. The Proof of Theorem 2

Proof. To prove Theorem 2, it is enough to show that

$$\hat{S} - \bar{S} \xrightarrow{p} 0. \tag{5.14}$$

Firstly, represent $\hat{S}_1$ as

$$\hat{S}_1 = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-\hat{W}_n(Z_i)}\right]\left[\frac{1}{n}\sum_{j=1}^{n}\frac{Z_j\delta_j}{1-\hat{W}_n(Z_j)}\right] - \frac{1}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{Z_i I(Z_i \ge a)\delta_i}{\left(1-\hat{W}_n(Z_i)\right)^2}\right] + (m-1)\frac{1}{n}\sum_{i=1}^{n}(Z_i - a)\frac{I(Z_i \ge a)\delta_i}{1-\hat{W}_n(Z_i)}. \tag{5.15}$$

Note that

$$\left|\frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-\hat{W}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-W(Z_i)}\right| \le \sup_{Z_i \le Z_{(n)}}\left|\frac{1}{1-\hat{W}_n(Z_i)} - \frac{1}{1-W(Z_i)}\right| \times \frac{1}{n}\sum_{i=1}^{n} I(Z_i \ge a)\delta_i \le P(X \ge a)\sup_{Z_i \le Z_{(n)}}\frac{\left|\hat{W}_n(Z_i) - W(Z_i)\right|}{\left|1-\hat{W}_n(Z_i)\right|\left|1-W(Z_i)\right|},$$

with probability 1,

$$\tag{5.16}$$

since

$$\frac{1}{n}\sum_{i=1}^{n} I(Z_i \ge a)\delta_i \to E\left[I(X_1 \ge a)\left(1-W(X_1)\right)\right] \le P(X \ge a), \tag{5.17}$$

with probability 1.

By Equation (5.16) and the following result (see [20]),

$$\sup_{Z_i \le Z_{(n)}}\left|\hat{W}_n(Z_i) - W(Z_i)\right| \xrightarrow{p} 0, \tag{5.18}$$

we know

$$\frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-\hat{W}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n}\frac{I(Z_i \ge a)\delta_i}{1-W(Z_i)} \xrightarrow{p} 0. \tag{5.19}$$

Similarly, we have

$$\frac{1}{n}\sum_{j=1}^{n}\frac{Z_j\delta_j}{1-\hat{W}_n(Z_j)} - \frac{1}{n}\sum_{j=1}^{n}\frac{Z_j\delta_j}{1-W(Z_j)} \xrightarrow{p} 0, \tag{5.20}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{Z_i I(Z_i \ge a)\delta_i}{\left[1-\hat{W}_n(Z_i)\right]^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{Z_i I(Z_i \ge a)\delta_i}{\left[1-W(Z_i)\right]^2} \xrightarrow{p} 0, \tag{5.21}$$

$$\frac{1}{n}\sum_{i=1}^{n}(Z_i - a)\frac{I(Z_i \ge a)\delta_i}{1-\hat{W}_n(Z_i)} - \frac{1}{n}\sum_{i=1}^{n}(Z_i - a)\frac{I(Z_i \ge a)\delta_i}{1-W(Z_i)} \xrightarrow{p} 0. \tag{5.22}$$

Combining Equations (5.15) with (5.19)-(5.22), we conclude that

$$\hat{S}_1 - \bar{S}_1 \xrightarrow{p} 0, \tag{5.23}$$

and

$$\hat{S}_2 - \bar{S}_2 \xrightarrow{p} 0. \tag{5.24}$$

Hence, Equation (5.14) has been proved. Together with Theorem 1's conclusion Theorem 2 holds.

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