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Half-Metric Spaces

Veeresh Kodekalmath [a*]

[a] *Department of Mathematics, REVA University, Bengaluru, Karnataka, India.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Short Research Article

Abstract

In this paper, I have introduced half-metric spaces on normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$, which are similar to metric spaces by relaxing a few conditions of metric space.I have also introduced even half-metric spaces,established some properties and discussed completeness in the context of half-metric space and even half-metric space.

Keywords: Metric Space; Half-Metric Space; Translation Invariant; Cauchy Sequences.

2010 Mathematics Subject Classification: 54E35, 54E50.

1 Introduction

Metric space is an ordered pair (M,d) where M is a non empty set and d is metric on M. [1] $d(x, y) : MXM \to \mathbb{R}$ such that for any $x, y, z \in M$ the following holds

$$1)d(x, y) \geq 0$$

$$2)d(x, y) = d(y, x)$$

$$3)d(x, z) \leq d(x, y) + d(y, z)$$

$$4)d(x, y) = 0 \iff x = y$$

A normed vector space or normed space is a vector space over the real or complex numbers, on which a norm is defined.A norm is a real-valued function defined on the vector space that is commonly

*Corresponding author: E-mail: mathveeresh168@gmail.com;

denoted [2] [3] [4] as $\|.\|$ which has the following properties:
for all vector x,y and scalar t

$$1)\|x\| \geq 0$$

$$2)\|x\| = 0 \iff x = 0$$

$$3)\|tx\| = |t|\|x\|$$

$$4)\|x + y\| \leq \|x\| + \|y\|.$$

A metric (M, d) on a linear space is said to be translation invariant if

$$d(x + a, y + a) = d(x, y)$$

$for\ all\ x, y, a \in M$[5].

A sequence $x_1, x_2, ...x_n$ is said be cauchy if for every postive real number $\epsilon > 0$ there is a positive Number N such that for all positive integers $m, n > N$, $d(x_m, x_n) < \epsilon$.

A metric space (M, d) is complete if every cauchy sequence in M converges in M [6] [7] [8].

2 Main Result

Definition 2.1. A half-metric space on a vector space equipped with $\|.\|$ over $\mathbb{R}$ or $\mathbb{C}$ is an ordered pair (M, d) ,where M is a non empty set,$\|.\|$ is the norm and d is half-metric on M,if the following holds,

$d(x, y) : MXM \to \mathbb{R}$ such that for any $x, y, z \in M$ the following holds

$$1)d(x, y) \geq 0$$

$$2)d(x, y) = d(y, x)$$

$$3)d(x, z) \leq d(x, y) + d(y, z)$$

$$4)d(0, y) = 0 \iff y = 0$$

2.1 Example

$(\mathbb{R}, d)$ where,$d(x, y) = |x^2 - y^2|$, clearly d is half-metric on $\mathbb{R}$.

Definition 2.2. A half-metric space is said to translation invariant if

$$d(x + a, y + a) = d(x, y)$$

$for\ all\ x, y, a \in M.$

Definition 2.3. A sequence $x_1, x_2, ...x_n$ is said be cauchy if for every postive real number $\epsilon > 0$ there is a positive Number N such that for all positive integers $m, n > N$, $d(x_m, x_n) < \epsilon$,where d is half-metric.

Definition 2.4. A half-metric space (M, d) is complete if every cauchy sequence in M converges in M.

Definition 2.5. A half-metric space is said to be even if for all x,y in M

$$d(x, y) = d(x, -y).$$

2.2 Note

Metric spaces on vector spaces cannot be even half-metric spaces because $d(x,x) = d(x,-x)$ only for $x = 0$ but metric spaces on vector spaces are half-metric spaces.

2.3 Example

$(\mathbb{R}, d)$, where ,$d(x,y) = |x^2 - y^2|$, consider $d(x,-y) = |x^2 - (-y)^2|$, $d(x,-y) = |x^2 - y^2| \implies d(x,-y) = d(x,y)$ for all x,y in $\mathbb{R}$.

Proposition 2.1. *If a half-metric space is even then it cannot be translation invariant.*

Alternatively if a half-metric space is translation invariant then it cannot be even.

Proof. let (M,d) be even half-metric space. then for all x,y in M

$$d(x,y) = d(x,-y)$$

assume d is translation invariant,then for all x,y,c in M

$$d(x+c, y+c) = d(x,y)$$

$$d(x+c, -y+c) = d(x,-y)$$

implies

$$d(x+c, y+c) = d(x+c, -y+c)$$

for all c in M, let $x = -c, y = c$

$$d(0,2c) = d(0,0)$$

$$d(0,2c) = 0$$

using properties of half-metric

$$2c = 0$$

$$c = 0$$

a contradiction because c was arbitrary, therefore even half-metric spaces cannot be translation invariant. □

2.4 Example

$(\mathbb{R}, d)$ *be a* $half - metric$ *space* such that $d(x,y) = |x| + |y|$, clearly it is an even half-metric space.
consider $d(1+2, 2+2) = d(3,4) = |3| + |4| = 7$
and $d(1,2) = |1| + |2| = 3$
$d(1+2, 2+2) \neq d(1,2)$ therefore d is not translation invariant.

2.5 Example

$(\mathbb{R}, d)$ *be a* $half - metric$ *space* such that, $d(x,y) = |x| + |y| + |x||y|$ clearly it is an even half-metric space on $\mathbb{R}$, therefore,it is not translation invariant.

2.6 Example

$(\mathbb{R}, d)$, $d = |x - y|$,clearly it is translation invariant,therefore it is not an even half-metric space.

Proposition 2.2. *An even half-metric space is complete* $\iff$ *all cauchy sequence in that space converge to zero.*

Proof. Let (M,d) be complete even half-metric space, let x_n be a cauchy sequence in M then

$$\lim_{n\to\infty} d(x_n, x) = 0$$

because M is complete, $\implies x \in M$, then by definition of even half-metric space

$$\lim_{n\to\infty} d(x_n, -x) = 0$$

$\implies x_n \to x$ as $n \to \infty$ and $x_n \to -x$ as $n \to \infty$

$$\implies x = 0$$

because limits are unique if they exist.

Conversely, let all cauchy sequence x_n converge to zero in (M, d), since M is even half-metric space, by definition of vector space, zero is in M. Hence (M, d) is complete. ☐

2.7 Example

$(\mathbb{R}, d)$ such that

$$d(x, y) = |x| + |y|$$

and x_n be a cauchy sequence in $\mathbb{R}$
i.e

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0$$

$\implies$

$$\lim_{n,m\to\infty} (|x_n| + |x_m|) = 0$$

$\implies$

$$x_n \to 0, x_m \to 0$$

as n,m $\to \infty$ therefore $(\mathbb{R}, d)$ is complete even half-metric space.

2.8 Example

$(\mathbb{R}, d)$, $d(x, y) = |x^2 - y^2|$, consider

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

x_n is cauchy in $\mathbb{R}$ because

$$d(\left(1 + \frac{1}{n}\right)^n, e) < \epsilon$$

for $n > N$, since d is even half -metric

$$d(\left(1 + \frac{1}{n}\right)^n, -e) < \epsilon$$

for $n > N$

$$\implies x_n \to e \textit{ and } x_n \to -e$$

which is not possible,therefore x_n is not convergent. $\implies (R, d)$ is not complete.

3 Conclusions

It is clear that all metric spaces on vector spaces are half-metric spaces but even half-metric spaces are not metric spaces on vector spaces.By relaxing a few conditions of metric spaces we obtain interesting results.In this article,I have defined half-metric spaces and even half-metric spaces for vector spaces,given plenty of examples,discussed completeness,translation invariant,cauchy sequences in context of half-metric spaces and even half-metric spcaces.

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Competing Interests

Author has declared that no competing interest exists.

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